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The Kähler Quotient Resolution of $\mathbb{C}^3/\Gamma$ singularities, the McKay correspondence and D=3 $\mathcal{N}=2$ Chern-Simons gauge theories

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Abstract

We advocate that the generalized Kronheimer construction of the Kähler quotient crepant resolution $\mathcal{M}_\zeta \longrightarrow \mathbb{C}^3/\Gamma$ of an orbifold singularity where $\Gamma \subset \mathrm{SU}(3)$ is a finite subgroup naturally defines the field content and the interaction structure of a superconformal Chern-Simons Gauge Theory. This latter is supposedly the dual of an M2-brane solution of $D=11$ supergravity with $\mathbb{C}\times\mathcal{M}_\zeta$ as transverse space. We illustrate and discuss many aspects of this type of constructions emphasizing that the equation $\boldsymbol{p}\wedge\boldsymbol{p}=0$ which provides the Kähler analogue of the holomorphic sector in the hyperKähler moment map equations canonically defines the structure of a universal superpotential in the CS theory. Furthermore the kernel $\mathscr{D}_\Gamma$ of the above equation can be described as the orbit with respect to a quiver Lie group $\mathscr{G}_\Gamma$ of a special locus $L_\Gamma \subset \mathrm{Hom}_\Gamma(\mathscr{Q}\otimes R, R)$ that has also a universal definition. We provide an extensive discussion of the relation between the coset manifold $\mathscr{G}_\Gamma/\mathscr{F}_\Gamma$, the gauge group $\mathscr{F}_\Gamma$ being the maximal compact subgroup of the quiver group, the moment map equations and the first Chern classes of the so named tautological vector bundles that are in one-to-one correspondence with the nontrivial irreps of Γ. These first Chern classes are represented by (1,1)-forms on $\mathcal{M}_\zeta$ and provide a basis for the cohomology group $H^2(\mathcal{M}_\zeta)$. We also discuss the relation with conjugacy classes of Γ and we provide the explicit construction of several examples emphasizing the role of a generalized McKay correspondence. The case of the ALE manifold resolution of $\mathbb{C}^2/\Gamma$ singularities is utilized as a comparison term and new formulae related with the complex presentation of Gibbons-Hawking metrics are exhibited.

Contents

1 Introduction

The issue of of $D = 3$ $\mathcal{N} = 2$ Chern-Simons gauge theories was reconsidered by one of us (P.F.) and P.A. Grassi from two complementary viewpoints [1]:

a) The constructive viewpoint of their formulation in terms of Integral Forms in superspace that brings together, within a unified (super)cohomological scheme, all the existing formulation of supersymmetric field theories, namely the component approach, the rheonomic approach and the superspace approach.

b) The field content and the interaction structure of the theories that are supposed to represent gauge duals of M2-brane solutions of $D = 11$ supergravity having a $\mathbb{C}^3/\Gamma$ quotient singularity in their transverse space. Such a situation, as we recall in the present paper, arises by considering the classical case of an $\mathrm{AdS}_4 \times \mathbb{S}^7$ solution where $\mathbb{S}^7$, viewed as a Hopf-fibration over $\mathbb{P}^3$, is modded by a discrete subgroup of $\Gamma \subset \mathrm{SU}(3) \subset \mathrm{SU}(4)$.

The physical motivations and the perspectives of the study encompassed in point b) have been extensively discussed in [1]. Here we are more concerned with the mathematical aspects of these theories and we aim at proving the following:

Statement 1.1 *There is a one-to-one map between the field-content and the interaction structure of a $D = 3$, $\mathcal{N} = 2$ Chern-Simons gauge theory and the generalized Kronheimer algorithm of solving quotient singularities $\mathbb{C}^3/\Gamma$ via a Kähler quotient based on the McKay correspondence. All items on both sides of the one-to-one correspondence are completely determined by the structure of the finite group Γ and by its specific embedding into* $\mathrm{SU}(3)$.

An ultra short summary of the results that we are going to present is the following. From the field–theoretic side the essential items defining the theory are:

1. The Kähler manifold $\mathscr{S}$ spanned by the Wess-Zumino multiplets. This is the $3|\Gamma|$ dimensional manifold $\mathscr{S}_\Gamma = \mathrm{Hom}_\Gamma(\mathscr{Q} \otimes R, R)$ where $\mathscr{Q}$ is the representation of Γ inside $\mathrm{SU}(3)$ and R denotes the regular representation of the discrete group.

2. The gauge group $\mathscr{F}_\Gamma$. This latter is identified as the maximal compact subgroup $\mathscr{F}_\Gamma$ of the complex quiver group $\mathscr{G}_\Gamma$ of complex dimension $|\Gamma| - 1$, to be discussed later. Since the action of $\mathscr{F}_\Gamma$ on $\mathscr{S}_\Gamma$ is defined by construction, the gauge interactions of the Wess-Zumino multiplets are also fixed and the associated moment maps are equally uniquely determined.

3. The Fayet-Iliopoulos parameters. These are in a one-to-one association with the ζ levels of the moment maps corresponding to the center of the gauge Lie algebra $\mathfrak{z}[\mathbb{F}_\Gamma]$[1]. The dimension of this center is r which is the number of nontrivial conjugacy classes of the discrete group Γ and also of its nontrivial irreps.

4. The superpotential $\mathscr{W}_\Gamma$. This latter is a cubic function uniquely associated with a quadratic constraint $[A,B] = [B,C] = [C,A] = 0$ which characterizes the generalized Kronheimer construction, defines a Kähler subvariety $\mathbb{V}_{|\Gamma|+2} \subset \mathscr{S}_\Gamma$ and admits a universal group theoretical description in terms of the quiver group $\mathscr{G}_\Gamma$.

5. In presence of all the above items the manifold of vacua of the gauge theory, namely of extrema at zero of its scalar potential, is just the minimal crepant resolution of the singularity $\mathscr{M}_\Gamma \to \frac{\mathbb{C}^3}{\Gamma}$, obtained as Kähler quotient of $\mathbb{V}_{|\Gamma|+2}$ with respect to the gauge group $\mathscr{F}_\Gamma$.

[1]In this paper we follow the convention that the names of the Lie groups are denoted with calligraphic letters $\mathscr{F}, \mathscr{G}, \mathscr{U}$, the corresponding Lie algebras being denoted by mathbb letters $\mathbb{F}, \mathbb{G}, \mathbb{U}$.

6. The Dolbeault cohomology of the space of vacua $\mathscr{M}_\Gamma$ is predicted by the finite group Γ structure in terms of a grading of its conjugacy classes named *age grading*. The construction of the homology cycles and exceptional divisors that are Poincaré dual to such cohomology classes is the most exciting issue in the present list of *geometry* $\Longleftrightarrow$ *field theory* correspondences, liable to give rise to many interesting physical applications. In this context a central item is the notion of tautological bundles associated with the nontrivial irreps of Γ that we discuss at length in the sequel [2].

2 The general form of $\mathscr{N}=2$ Chern-Simons Gauge Theories

To discuss the one-to-one map in statement 1.1, we begin by recalling, from the results of [1], the general structure of the component lagrangian for an $\mathscr{N}=2$, $D=3$ Chern-Simons gauge theory. The results of [1] correspond to a generalization and more geometrical transcription of the general form of coupling of matter with D=3 gauge theories constructed in 1999 in a series of three papers [3–5], using auxiliary fields and the rheonomic approach. In 1999, the motivation to consider this type of theories was the reinterpretation [3–8], within the AdS/CFT scheme of the Kaluza-Klein spectra of supergravity localized on backgrounds $\mathrm{AdS}_4\times\mathscr{M}_7$ that were calculated in the years 1982-1985 [9–25]. In all those cases the manifold $\mathscr{M}_7$ was smooth, actually a Sasakian coset manifold. As we better explain, in this paper we are interested to apply the same ideas to the case where the metric cone ove $\mathscr{M}_7$ is an orbifold $\mathbb{C}\times\mathbb{C}^3/\Gamma$, denoting by Γ a finite subgroup of $\mathrm{SU}(3)$.

The lagrangian of $\mathscr{N}=2$ Chern-Simons Gauge Theory, as systematized in [1], takes the following form:

$$
\begin{aligned}
\mathscr{L}_{CSoff} \;=\;& -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A}+\frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right)+\left(\frac{1}{2}g_{ij^\star}\Pi^{m|i}\nabla\overline{z}^{j^\star}+\overline{\Pi}^{m|j^\star}\nabla z^i\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
&-\frac{1}{6}g_{ij^\star}\Pi^{m|i}\overline{\Pi}^{m|j^\star}e^r\wedge e^s\wedge e^t\,\varepsilon_{rst}\\
&+\mathrm{i}\frac{1}{2}g_{ij^\star}\left(\overline{\chi}^{j^\star}\gamma^n\nabla\chi^i+\overline{\chi}^i_c\gamma^n\nabla\chi^{i^\star}_c\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
&\Bigg(-\frac{1}{3}M^\Lambda\left(\partial_i k^j_\Lambda g_{j\ell^\star}\overline{\chi}^{\ell^\star}\chi^i+\partial_{i^\star}k^{j^\star}_\Lambda g_{j\ell^\star}\overline{\chi}^\ell_c\chi^{i^\star}_c\right)+\frac{\alpha}{3}\left(\overline{\lambda}^\Lambda\lambda^\Sigma+\overline{\lambda}^\Lambda_c\lambda^\Sigma_c\right)\kappa_{\Lambda\Sigma}\\
&+\mathrm{i}\frac{1}{3}\left(\overline{\chi}^{j^\star}_c\lambda^\Lambda k^i_\Lambda-\overline{\chi}^i_c\lambda^\Lambda k^{j^\star}_\Lambda\right)g_{ij^\star}\\
&+\frac{1}{6}\left(\partial_i\partial_j\mathscr{W}\,\overline{\chi}^i_c\chi^j+\partial_{i^\star}\partial_{j^\star}\overline{\mathscr{W}}\,\overline{\chi}^{i^\star}\chi^{j^\star}_c\right)\Bigg)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
&-V\left(M,D,\mathscr{H},z,\overline{z}\right)\varepsilon_{mnp}\,e^m\wedge e^n\wedge e^p
\end{aligned}
\tag{2.1}
$$

where:

1. The complex scalar fields z^i span a Kähler manifold $\mathscr{S}$, $g_{ij^\star}$ denoting its Kähler metric.
2. $\Pi^{m|i}$ are auxiliary fields that are identified with the world volume derivatives of the scalar z^i by their own equation of motion.
3. The one–forms e^m denote the dreiben of the world volume.
4. $\mathscr{A}^\Lambda$ is the gauge-one form of the gauge group $\mathscr{F}_\Gamma$.
5. λ^Λ are the gauginos, namely the spin $\frac{1}{2}$ partners of the gauge bosons $\mathscr{A}^\Lambda$
6. χ^i are the chiralinos, namely the spin $\frac{1}{2}$ partners of the Wess-Zumino scalars z^i.
7. M^Λ are the real scalar fields in the adjoint of the gauge group that complete the gauge multiplet together with the gauginos and the gauge bosons.

8. $\mathscr{W}(z)$ is the superpotential.

9. k^i_Λ are the Killing vectors of the Kähler metric of S, associated with the generators of the gauge group.

10. $\kappa_{\Lambda\Sigma}$, made of constants denotes the Cartan Killing metric on the Lie algebra $\mathbb{F}_\Gamma$ of the gauge group $\mathscr{F}_\Gamma$

The scalar potential in terms of physical and auxiliary fields is the following one:

$$
\begin{aligned}
V(M,D,\mathscr{H},z,\bar{z}) &= \left(\frac{\alpha}{3} M^\Lambda \kappa_{\Lambda\Sigma} - \frac{1}{6}\mathscr{P}_\Sigma(z,\bar{z}) + \frac{1}{6}\zeta_I \mathfrak{C}^I_\Sigma\right) D^\Sigma + \frac{1}{6} M^\Lambda M^\Sigma k^i_\Lambda k^{j^\star}_\Sigma g_{ij^\star} \\
&\quad + \frac{1}{6}\left(\mathscr{H}^i \partial_i \mathscr{W} + \mathscr{H}^{\ell^\star} \partial_{\ell^\star} \overline{\mathscr{W}}\right) - \frac{1}{6} g_{i\ell^\star} \mathscr{H}^i \mathscr{H}^{\ell^\star}
\end{aligned}
\tag{2.2}
$$

where $\mathscr{P}_\Sigma(z,\bar{z})$ are the moment maps associated with each generator of the gauge-group, ζ_I are the Fayet-Iliopoulos parameters associated with each generator of the center of the gauge Lie algebra $\mathfrak{z}(\mathbb{F}_\Gamma)$, $\mathscr{H}^i$ are the complex auxiliary fields of the Wess-Zumino multiplets and D^Λ are the auxiliary scalars of the vector multiplets. By $\mathfrak{C}^I_\Sigma$ we denote the projector onto a basis of generators of the Lie Algebra center $\mathfrak{z}[\mathbb{F}_\Gamma]$.

In these theories the gauge multiplet does not propagate and it is essentially made of lagrangian multipliers for certain constraints. Indeed the auxiliary fields, the gauginos and the vector multiplet scalars have algebraic field equations so that they can be eliminated by solving such equations of motion. The vector multiplet auxiliary scalars D^Λ appear only as lagrangian multipliers of the constraint[2]:

$$
M^\Lambda = \frac{1}{2\alpha} \kappa^{\Lambda\Sigma} \left(\mathscr{P}_\Sigma - \zeta_I \mathfrak{C}^I_\Sigma\right)
\tag{2.3}
$$

while the variation of the auxiliary fields $\mathscr{H}^{j^\star}$ of the Wess Zumino multiplets yields:

$$
\mathscr{H}^i = g^{ij^\star} \partial_{j^\star} \overline{W} \quad ; \quad \overline{\mathscr{H}}^{j^\star} = g^{ij^\star} \partial_i W
\tag{2.4}
$$

On the other hand, the equation of motion of the field M^Λ implies:

$$
D^\Lambda = -\frac{1}{\alpha} \kappa^{\Lambda\Gamma} g_{ij^\star} k^i_\Gamma k^{j^\star}_\Sigma M^\Sigma = -\frac{1}{2\alpha^2} g_{ij^\star} \kappa^{\Lambda\Gamma} k^i_\Gamma k^{j^\star}_\Sigma \kappa^{\Sigma\Delta} \left(\mathscr{P}_\Delta - \zeta_I \mathfrak{C}^I_\Delta\right)
\tag{2.5}
$$

which finally resolves all the auxiliary fields in terms of functions of the physical scalars.

Upon use of both constraints (2.3) and (2.4) the scalar potential takes the following positive definite form:

$$
\begin{aligned}
V(z,\bar{z}) &= \frac{1}{6}\left(\partial_i \mathscr{W} \partial_{j^\star} \overline{\mathscr{W}} g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma} \left(\mathscr{P}_\Lambda - \zeta_I \mathfrak{C}^I_\Lambda\right)\left(\mathscr{P}_\Sigma - \zeta_J \mathfrak{C}^J_\Sigma\right)\right) \\
\mathbf{m}^{\Lambda\Sigma}(z,\bar{z}) &\equiv \frac{1}{4\alpha^2} \kappa^{\Lambda\Gamma} \kappa^{\Sigma\Delta} k^i_\Gamma k^{j^\star}_\Delta g_{ij^\star}
\end{aligned}
\tag{2.6}
$$

In a similar way the gauginos can be resolved in terms of the chiralinos:

$$
\lambda^\Lambda = -\frac{1}{2\alpha} \kappa^{\Lambda\Sigma} g_{ij^\star} \chi^i k^{j^\star}_\Sigma \quad ; \quad \lambda^\Lambda_c = -\frac{1}{2\alpha} \kappa^{\Lambda\Sigma} g_{ij^\star} \chi^{j^\star} k^i_\Sigma
\tag{2.7}
$$

In this way if we were able to eliminate also the gauge one form $\mathscr{A}$ the Chern-Simons gauge theory would reduce to a theory of Wess-Zumino multiplets with additional interactions. The elimination of $\mathscr{A}$, however, is not possible in the nonabelian case and it is possible in the abelian case only through duality nonlocal transformations. This is the corner where interesting nonperturbative dynamics is hidden.

[2]As it is customary for all metrics $\kappa^{\Lambda\Sigma}$ with upper indices denotes the inverse of the Cartan Killing metric $\kappa_{\Delta\Pi}$ with lower indices.

2.1 A special class of $\mathscr{N}=2$ Chern-Simons gauge theory in three dimensions

In the realization of the one-to-one map advocated in statement 1.1, we are interested in theories where the Wess-Zumino multiplets are identified with the non-vanishing entries of a triple of matrices, named A,B,C, and the superpotential takes the following form:

$$\begin{aligned}\mathscr{W} &= \text{const} \times \text{Tr}(A[B,C]) + \text{Tr}(B[C,A]) + \text{Tr}(C[A,B]) \\ &= 3\,\text{const} \times (\text{Tr}(ABC) - \text{Tr}(ACB))\end{aligned} \tag{2.8}$$

Because of the positive-definiteness of the Kähler metric $g^{ij^\star}$ and of the Killing metric $\mathbf{m}^{\Lambda\Sigma}$ the zero of the potential, namely the vacua, are characterized by the two conditions:

$$\partial_i \mathscr{W} = 0 \quad \Rightarrow [A,B] = [B,C] = [C,A] = 0 \tag{2.9}$$

$$\mathscr{P}_\Lambda = \zeta_I \mathfrak{C}^I_\Lambda \tag{2.10}$$

which will have a distinctive interpretation in the Kähler quotient construction à la Kronheimer. Notice that $\mathfrak{C}^I_\Lambda$ denotes the projector of the gauge Lie algebra onto its center, as we already said.

3 On superconformal Chern-Simons theories dual to M2-branes

In this short section we collect some issues and hints relative to the construction of superconformal gauge theories dual to orbifolds of the M2-brane transverse space with respect to $\Gamma \subset \text{SU}(3) \subset \text{SU}(4)$. The most important conclusion is that, as long as we require the existence of a complex structure of $\mathbb{R}^8$ compatible with L_{168}[3], we reduce the singularity to:

$$\frac{\mathbb{C}^4}{\Gamma} \to \mathbb{C} \times \frac{\mathbb{C}^3}{\Gamma} \quad ; \quad \Gamma \subset \text{SU}(3) \tag{3.1}$$

From the point of view of supergravity and string theory the factorization of a $\mathbb{C}$ factor is welcome. It provides the means to reduce M2-branes to D2-branes in $D=10$ type IIA theory.

As it is well known SO(8)-gauged supergravity is obtained from d=11 supergravity compactified on:

$$\text{AdS}_4 \times \mathbb{S}^7 \tag{3.2}$$

which is the near horizon geometry of an M2-brane with $\mathbb{R}^8$ transverse space. Indeed $\mathbb{R}^8$ is the metric cone on $\mathbb{S}^7$. The entire Kaluza-Klein spectrum which constitutes the spectrum of BPS operators of the d=3 theory is organized in short representations of the supergroup:

$$\text{Osp}(8|4) \tag{3.3}$$

Our discussion leads to the conclusion that we can consider the compactification of supergravity on orbifolds of the following type:

$$\mathbb{C}_\Gamma \mathbb{S}^7 = \frac{\mathbb{S}^7}{\Gamma} \quad ; \quad \Gamma \subset \text{SU}(3) \subset \text{SO}(8) \tag{3.4}$$

The corresponding M2-brane solution has the orbifold:

$$\mathbb{C}_\Gamma \mathbb{R}^8 = \frac{\mathbb{R}^8}{\Gamma} \tag{3.5}$$

[3] Following the notations of [26] by L_{168} we denote the simple group of order 168 that is isomorphic with $\text{PSL}(2,7)$ and which is also the largest non abelian non solvable finite subgroup of SU(3), according with the classification of [27, 28].

as transverse space.

The massive and massless modes of the Kalauza Klein spectrum are easily worked out from the $\mathrm{Osp}(8|4)$ spectrum of the 7-sphere. Indeed since the group Γ is embedded by the above construction into $\mathrm{SO}(8) \subset \mathrm{Osp}(8|4)$, it suffices to cut the spectrum to the Γ singlets.

3.1 Quotient singularities and CS gauge groups

What we said above can be summarized by saying that we construct models where M2-branes are probing the singularity (3.5) and we might be interested in the smooth manifold obtained by blowing up the latter.

From another viewpoint, according to [29] the superconformal Chern-Simons theory with the gauge groups $\mathrm{SU(N)} \times \mathrm{SU(N)}$ at level k is dual to supergravity on:

$$\mathrm{AdS}_4 \times \frac{\mathbb{S}^7}{\mathbb{Z}_k} \tag{3.6}$$

namely the association is at the orbifold point between the gauge-group structure and the Γ discrete group, so that:

$$\mathrm{SU(N)} \mid_{k \text{ level}} \times \mathrm{SU(N)} \mid_{k \text{ level}} \Leftrightarrow \mathbb{Z}_k \tag{3.7}$$

Despite the difficulties in working out the blowup, it is therefore legitimate to ask the question: what is the CS gauge group corresponding to $\Gamma \subset \mathrm{SU}(3)$:

$$\text{gauge CS ?} \Leftrightarrow \Gamma \subset \mathrm{SU}(3) \tag{3.8}$$

As it is implicit in the comparison between eq. (3.8) and eq. (3.7), the statement contained in [29] is incomplete. It is not sufficient to say $\frac{\mathbb{S}^7}{\mathbb{Z}_k}$. In order to derive the dual superconformal field theory it is essential to specify the embedding of $\mathbb{Z}_k$ into the isometry group $\mathrm{SO}(8)$ of the seven sphere. Different embeddings of the same discrete group can lead to different CS gauge theories.

3.2 Some suggestions from ALE manifolds

Some useful suggestions on this conceptual link can arise by comparing with the case of well known singularities like $\mathbb{C}^2/\Gamma$, the discrete group Γ being one of the finite subgroups of $\mathrm{SU}(2)$ falling into the ADE classification. In that case the blowup of the singularities can be done by means of a hyperKähler quotient according to the Kronheimer construction [30, 31]. The gauge group is essentially to be identified with the nonabelian extension $\mathrm{U}(1) \to \mathrm{U(N)}$ of the group $\mathscr{F}$ one utilizes in the hyperKähler quotient. The group $\mathscr{F}_\Gamma$ is a product of $\mathrm{U}(1)$'s as long as the discrete group Γ is the cyclic group $\mathbb{Z}_k$, yet it becomes nonabelian with factors SU(k) when Γ is nonabelian. As far as we know no one has constructed CS gauge theories corresponding to M2-branes that probe singularities of the type $\mathbb{C}^2 \times \mathbb{C}^2/\Gamma$ with a nonabelian Γ. This case might be a ground-zero case to investigate.

3.3 Moduli of the blowup and superconformal operators: inspirations from geometry

It must be stressed that in the spectrum of the conformal field theory obtained at the orbifold point (which corresponds to the Kaluza Klein spectrum in the case of smooth manifolds) there must be those associated with the moduli of the blowup. By similarity with the case of superstrings at orbifold points, we expect that these are *twisted states*, namely states not visible in the Kaluza Klein spectrum on the orbifold. Yet when the orbifold is substituted with its smooth counterpart, obtained blowing of up the singularity, these states should appear as normal states in the supergravity Kaluza Klein spectrum.

3.4 Temporary conclusion

The above discussion shows that the embedding of Γ is fundamental. We can treat $L_{168} \equiv \mathrm{PSL}(2,\mathbb{Z}_7)$ and other discrete group Γ singularities when Γ has a holomorphic action on $\mathbb{C}^4$. This happens when

$$\Gamma \subset \mathrm{SU}(3) \subset \mathrm{SO}(8) \tag{3.9}$$

In this case the singularity is just $\mathbb{C} \times \mathbb{C}^3/\Gamma$ and $\mathbb{C}^3/L_{168}$ is the blowup described by Markushevich in [32]. We mention it again in a later section. The classification of discrete subgroups of $\mathrm{SL}(3,\mathbb{C})$ was achieved at the dawn of the XXth century in [27, 28]. The largest nonabelian nontrivial group appearing in this classification is the unique simple group with 168 elements named L_{168} (see [26] and [33] for a thorough discussion). The other possibilities are provided by a finite list of cyclic and solvable groups reviewed for instance in [34].

3.5 Quotient singularities and M2-branes

We come now to the mathematics which is of greatest interest to us in order to address the physical problem at stake, *i.e.*, the construction of CS theories dual to M2-branes that have the metric cone on orbifolds $\mathbb{S}^7/\Gamma$ as transverse space. The first step is to show that such metric cone is just $\mathbb{C}^4/\Gamma$. This is a rather simple fact but it is of the utmost relevance since it constitutes the very bridge between the mathematics of quotient singularities, together with their resolutions, and the physics of CS theories. The pivot of this bridge is the complex Hopf fibration of the 7-sphere. The argument leading to the above conclusion was provided in the paper [1] and we do not deem it necessary to repeat it here. We just jump to the conclusion there reached. The space $\mathbb{C}^4 - \{0\}$ can be regarded as the total space of the canonical $\mathbb{C}^\star$-fibration over $\mathbb{CP}^3$:

$$\begin{aligned} \pi &: \mathbb{C}^4 - \{0\} \to \mathbb{CP}^3 \\ \forall y \in \mathbb{CP}^3 &: \pi^{-1}(y) \sim \mathbb{C}^\star \end{aligned} \tag{3.10}$$

By restricting to the unit sphere in $\mathbb{C}^4$ we obtain the Hopf fibration of the seven sphere:

$$\begin{aligned} \pi &: \mathbb{S}^7 \to \mathbb{CP}^3 \\ \forall y \in \mathbb{CP}^3 &: \pi^{-1}(y) \sim \mathbb{S}^1 \end{aligned} \tag{3.11}$$

The consequence of such a discussion is that if we have a finite subgroup $\Gamma \subset \mathrm{SU}(4)$, which obviously is an isometry of $\mathbb{CP}^3$, we can consider its action both on $\mathbb{CP}^3$ and on the seven sphere so that we have:

$$\mathrm{AdS}_4 \times \frac{\mathbb{S}^7}{\Gamma} \to \partial\mathrm{AdS}_4 \times \frac{\mathbb{C}^4}{\Gamma} \tag{3.12}$$

We are therefore interested in describing the theory of M2-branes probing the singularity $\frac{\mathbb{C}^4}{\Gamma}$. Hence an important guiding line in addressing mathematical questions comes from their final use in connection with M2-brane solutions of $D = 11$ supergravity and with the construction of quantum gauge theories dual to such M2-solutions of supergravity.

Recalling the results of [1] we start from the following diagram

$$K_3 \xleftarrow{\pi} \mathscr{M}_7 \overset{Cone}{\hookrightarrow} K_4 \overset{\mathscr{A}}{\hookrightarrow} \mathbb{V}_q \tag{3.13}$$

where $\mathscr{M}_7$ is the compact manifold on which D=11 supergravity is compactified and $\mathbb{V}_q$ denotes some appropriate algebraic variety of complex dimension q. It is required that $\mathscr{M}_7$ should be a Sasakian manifold.

What Sasakian means is visually summarized in the following table.

base of the fibration	projection	7-manifold	inclusion	metric cone
$\mathscr{B}_6$	$\stackrel{\pi}{\longleftarrow}$	$\mathscr{M}_7$	$\hookrightarrow$	$\mathscr{C}(\mathscr{M}_7)$
$\Updownarrow$	$\forall p \in \mathscr{B}_6 \quad \pi^{-1}(p) \sim \mathbb{S}^1$	$\Updownarrow$		$\Updownarrow$
Kähler K_3		Sasakian		Kähler Ricci flat K_4

First of all the $\mathscr{M}_7$ manifold must admit an $\mathbb{S}^1$-fibration over a Kähler three-fold K_3:

$$\pi \quad : \quad \mathscr{M}_7 \stackrel{\mathbb{S}^1}{\longrightarrow} K_3 \tag{3.14}$$

Calling z^i the three complex coordinates of K_3 and ϕ the angle spanning $\mathbb{S}^1$, the fibration means that the metric of $\mathscr{M}_7$ admits the following representation:

$$ds^2_{\mathscr{M}_7} = (d\phi - \mathscr{A})^2 + g_{ij^\star}\, dz^i \otimes d\bar{z}^{j^\star} \tag{3.15}$$

where the one–form $\mathscr{A}$ is some suitable connection one–form on the U(1)-bundle (3.14).

Secondly the metric cone $\mathscr{C}(\mathscr{M}_7)$ over the manifold $\mathscr{M}_7$ defined by the direct product $\mathbb{R}_+ \times \mathscr{M}_7$ equipped with the following metric:

$$ds^2_{\mathscr{C}(\mathscr{M}_7)} = dr^2 + 4\,e^2\,r^2\,ds^2_{\mathscr{M}_7} \tag{3.16}$$

should also be a Ricci-flat complex Kähler 4-fold. In the above equation e just denotes a constant scale parameter with the dimensions of an inverse length $[e] = \ell^{-1}$.

Altogether the Ricci flat Kähler manifold K_4, which plays the role of transverse space to the M2-branes, is a line bundle over the base manifold K_3:

$$\begin{array}{rcl} \pi & : & K_4 \longrightarrow K_3 \\ \forall p \in K_3 & & \pi^{-1}(p) \sim \mathbb{C} \end{array} \tag{3.17}$$

In [1], following [7], it was emphasized that the fundamental geometrical clue to the field content of the *superconformal gauge theory* on the boundary is provided by the construction of the Kähler manifold K_4 as a holomorphic algebraic variety in some higher dimensional affine or projective space $\mathbb{V}_q$, plus a Kähler quotient. The equations identifying the algebraic locus in $\mathbb{V}_q$ are related with the superpotential $\mathscr{W}$ appearing in the $d = 3$ lagrangian, while the Kähler quotient is related with the D-terms appearing in the same lagrangian. The coordinates u^α of the space $\mathbb{V}_q$ are the scalar fields of the *superconformal gauge theory*, whose vacua, namely the set of extrema of its scalar potential, should be in a one–to–one correspondence with the points of K_4. Going from one to multiple M2–branes just means that the coordinates z^i of $\mathbb{V}_q$ acquire color indices under a proper set of color gauge groups and are turned into matrices. In this way we obtain *quivers*.

This is the main link between the D=3 Chern-Simons gauge theories discussed in sections 2, 2.1 and the geometry of the transverse space to the branes.

Next in [1] eq. (3.13) was rewritten in slightly more general terms. The AdS$_4$ compactification of $D = 11$ supergravity is obtained by utilizing as complementary 7-dimensional space a manifold $\mathscr{M}_7$ which occupies the above displayed position in the inclusion–projection diagram (3.13). The metric cone $\mathscr{C}(\mathscr{M}_7)$ enters the game when, instead of looking at the vacuum:

$$\mathrm{AdS}_4 \otimes \mathscr{M}_7 \tag{3.18}$$

we consider the more general M2-brane solutions of D=11 supergravity, where the D=11 metric is of the following form:

$$ds^2_{11} = H(y)^{-\frac{2}{3}} \left(d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}\right) - H(y)^{\frac{1}{3}} \left(ds^2_{\mathscr{M}_8}\right) \tag{3.19}$$

$\eta_{\mu\nu}$ being the constant Lorentz metric of $\text{Mink}_{1,2}$ and:

$$ds^2_{\mathscr{M}_8} = dy^I \otimes dy^J g_{IJ}(y) \tag{3.20}$$

being a Ricci-flat metric on an asymptotically locally Euclidean 8-manifold $\mathscr{M}_8$. In eq. (3.19) the symbol $H(y)$ denotes a harmonic function over the manifold $\mathscr{M}_8$, namely:

$$\Box_g H(y) = 0 \tag{3.21}$$

Eq. (3.21) is the only differential constraint required in order to satisfy all the field equations of $D = 11$ supergravity in presence of the standard M2-brain ansatz for the 3-form field:

$$\mathbf{A}^{[3]} \propto H(y)^{-1} \left(d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho \, \varepsilon_{\mu\nu\rho}\right) \tag{3.22}$$

In this more general setup the manifold $\mathscr{M}_8$ is what substitutes the metric cone $\mathscr{C}(\mathscr{M}_7)$. To see the connection between the two viewpoints it suffices to introduce the radial coordinate $r(y)$ by means of the position:

$$H(y) = 1 - \frac{1}{r(y)^6} \tag{3.23}$$

The asymptotic region where $\mathscr{M}_8$ is required to be locally Euclidean is defined by the condition $r(y) \to \infty$. In this limit the metric (3.20) should approach the flat Euclidean metric of $\mathbb{R}^8 \simeq \mathbb{C}^4$. The opposite limit where $r(y) \to 0$ defines the near horizon region of the M2-brane solution. In this region the metric (3.19) approaches that of the space (3.18), the manifold $\mathscr{M}_7$ being a codimension one submanifold of $\mathscr{M}_8$ defined by the limit $r \to 0$.

To be mathematically more precise let us consider the harmonic function as a map:

$$\mathfrak{H} \quad : \quad \mathscr{M}_8 \to \mathbb{R}_+ \tag{3.24}$$

This view point introduces a foliation of $\mathscr{M}_8$ into a one-parameter family of 7-manifolds:

$$\forall h \in \mathbb{R}_+ \quad : \quad \mathscr{M}_7(h) \equiv \mathfrak{H}^{-1}(h) \subset \mathscr{M}_8 \tag{3.25}$$

In order to have the possibility of residual supersymmetries we are interested in cases where the Ricci flat manifold $\mathscr{M}_8$ is actually a Ricci-flat Kähler 4-fold.

In this way the appropriate rewriting of eq. (3.13) is as follows:

$$K_3 \xleftarrow{\pi} \mathscr{M}_7 \quad \overset{\mathfrak{H}^{-1}}{\longleftarrow} \quad K_4 \quad \overset{\mathscr{A}}{\hookrightarrow} \quad \mathbb{V}_q \tag{3.26}$$

Next we recall the general pattern laid down in [1] that will be our starting point.

The $\mathscr{N} = 8$ case with no singularities. The prototype of the above inclusion–projection diagram is provided by the case of the M2-brane solution with all preserved supersymmetries. In this case we have:

$$\mathbb{CP}^3 \quad \xleftarrow{\pi} \quad \mathbb{S}^7 \quad \overset{Cone}{\hookrightarrow} \quad \mathbb{C}^4 \quad \overset{\mathscr{A}=\mathrm{Id}}{\hookrightarrow} \quad \mathbb{C}^4 \tag{3.27}$$

On the left we just have the projection map of the Hopf fibration of the 7-sphere. On the right we have the inclusion map of the 7 sphere in its metric cone $\mathscr{C}(\mathbb{S}^7) \equiv \mathbb{R}^8 \sim \mathbb{C}^4$. The last algebraic inclusion map is just the identity map, since the algebraic variety $\mathbb{C}^4$ is already smooth and flat and needs no extra treatment.

The singular orbifold cases. The next orbifold cases are those of interest to us in this paper. Let $\Gamma \subset \mathrm{SU}(4)$ be a finite discrete subgroup of $\mathrm{SU}(4)$. Then eq. (3.27) is replaced by the following one:

$$\frac{\mathbb{CP}^3}{\Gamma} \quad \stackrel{\pi}{\longleftarrow} \quad \frac{\mathbb{S}^7}{\Gamma} \quad \stackrel{Cone}{\hookrightarrow} \quad \frac{\mathbb{C}^4}{\Gamma} \quad \stackrel{\mathscr{A}=?}{\hookrightarrow} \quad ? \tag{3.28}$$

The consistency of the above quotient is guaranteed by the inclusion $\mathrm{SU}(4) \subset \mathrm{SO}(8)$. The question marks can be removed only by separating the two cases:

A) Case: $\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SU}(2)_{\mathrm{I}} \otimes \mathrm{SU}(2)_{\mathrm{II}} \subset \mathrm{SU}(4)$. Here we obtain:

$$\frac{\mathbb{C}^4}{\Gamma} \simeq \mathbb{C}^2 \times \frac{\mathbb{C}^2}{\Gamma} \tag{3.29}$$

and everything is under full control for the Kleinian $\frac{\mathbb{C}^2}{\Gamma}$ singularities and their resolution à la Kronheimer in terms of hyperKähler quotients.

B) Case: $\Gamma \subset \mathrm{SU}(3) \subset \mathrm{SU}(4)$. Here we obtain:

$$\frac{\mathbb{C}^4}{\Gamma} \simeq \mathbb{C} \times \frac{\mathbb{C}^3}{\Gamma} \tag{3.30}$$

and the study and resolution of the singularity $\frac{\mathbb{C}^3}{\Gamma}$ in a physicist friendly way is the main issue of the present paper. The comparison of case B) with the well known case A) will provide us with many important hints.

Let us begin by erasing the question marks in case A). Here we can write:

$$\frac{\mathbb{CP}^3}{\Gamma} \quad \stackrel{\pi}{\longleftarrow} \quad \frac{\mathbb{S}^7}{\Gamma} \quad \stackrel{Cone}{\hookrightarrow} \quad \mathbb{C}^2 \times \frac{\mathbb{C}^2}{\Gamma} \quad \stackrel{\mathrm{Id}\times\mathscr{A}_W}{\hookrightarrow} \quad \mathbb{C}^2 \times \mathbb{C}^3 \tag{3.31}$$

In the first inclusion map on the right, Id denotes the identity map $\mathbb{C}^2 \to \mathbb{C}^2$ while $\mathscr{A}_W$ denotes the inclusion of the orbifold $\frac{\mathbb{C}^2}{\Gamma}$ as a singular variety in $\mathbb{C}^3$ cut out by a single polynomial constraint:

$$\begin{aligned} \mathscr{A}_W \quad &: \quad \frac{\mathbb{C}^2}{\Gamma} \to \mathbf{V}(\mathscr{I}_\Gamma^W) \subset \mathbb{C}^3 \\ \mathbb{C}\left[\mathbf{V}(\mathscr{I}_\Gamma)\right] \quad &= \quad \frac{\mathbb{C}[u,w,z]}{W_\Gamma(u,w,z)} \end{aligned} \tag{3.32}$$

where by $\mathbb{C}\left[\mathbf{V}(\mathscr{I}_\Gamma)\right]$ we denote the *coordinate ring* of the algebraic variety $\mathbf{V}$. As we recall in more detail in the next section, the variables u,w,z are polynomial Γ-invariant functions of the coordinates z_1, z_2 on which Γ acts linearly. The unique generator $W_\Gamma(u,w,z)$ of the ideal $\mathscr{I}_\Gamma^W$ which cuts out the singular variety isomorphic to $\frac{\mathbb{C}^2}{\Gamma}$ is the unique algebraic relation existing among such invariants. In the next sections we discuss the relation between this algebraic equation and the embedding in higher dimensional algebraic varieties associated with the McKay quiver and the hyperKähler quotient.

Let us now consider the case B). Up to this level things go in a quite analogous way with respect to case A). Indeed we can write

$$\frac{\mathbb{CP}^3}{\Gamma} \quad \stackrel{\pi}{\longleftarrow} \quad \frac{\mathbb{S}^7}{\Gamma} \quad \stackrel{Cone}{\hookrightarrow} \quad \mathbb{C} \times \frac{\mathbb{C}^3}{\Gamma} \quad \stackrel{\mathrm{Id}\times\mathscr{A}_\mathscr{W}}{\hookrightarrow} \quad \mathbb{C} \times \mathbb{C}^4 \tag{3.33}$$

In the last inclusion map on the right, Id denotes the identity map $\mathbb{C} \to \mathbb{C}$ while $\mathscr{A}_\mathscr{W}$ denotes the inclusion of the orbifold $\frac{\mathbb{C}^3}{\Gamma}$ as a singular variety in $\mathbb{C}^4$ cut out by a single polynomial constraint:

$$\begin{array}{rcl} \mathscr{A}_{\mathscr{W}} & : & \frac{\mathbb{C}^3}{\Gamma} \to \mathbf{V}(\mathscr{I}_\Gamma) \subset \mathbb{C}^4 \\ \mathbb{C}\left[\mathbf{V}\left(\mathscr{I}_\Gamma\right)\right] & \sim & \frac{\mathbb{C}[u_1,u_2,u_3,u_4]}{\mathscr{W}_\Gamma(u_1,u_2,u_3,u_4)} \end{array} \tag{3.34}$$

Indeed as we show in later sections for the case $\Gamma = \mathrm{L}_{168}$, discussed by Markushevich, and for all of its subgroups[4], including $\Gamma = \mathrm{G}_{21} \subset \mathrm{L}_{168}$, the variables u_1,u_2,u_3,u_4 are polynomial Γ-invariant functions of the coordinates z_1,z_2,z_3 on which Γ acts linearly. The unique generator $\mathscr{W}_\Gamma(u_1,u_2,u_3,u_4)$ of the ideal $\mathscr{I}_\Gamma$ which cuts out the singular variety isomorphic to $\frac{\mathbb{C}^3}{\Gamma}$ is the unique algebraic relation existing among such invariants. As for the relation of this algebraic equation with the embedding in higher dimensional algebraic varieties associated with the McKay quiver, things are now more complicated.

In the years 1990s up to 2010s there has been an intense activity in the mathematical community on the issue of the crepant resolutions of $\mathbb{C}^3/\Gamma$ (see for instance [32, 35, 36, 36, 37]) that has gone on almost unnoticed by physicists since it was mostly formulated in the abstract language of algebraic geometry, providing few clues to the applicability of such results to gauge theories and branes. Yet, once translated into more explicit terms, by making use of coordinate patches, and equipped with some additional ingredients of Lie group character, these mathematical results turn out to be not only useful, but rather of outmost relevance for the physics of M2-branes. In the present paper we aim at drawing the consistent, systematic scheme which emerges in this context upon a proper interpretation of the mathematical constructions.

So let us consider the case of smooth resolutions. In case A) the smooth resolution is provided by a manifold ALE_Γ and we obtain the following diagram:

$$\mathscr{M}_7 \quad \stackrel{\mathfrak{H}^{-1}}{\longleftarrow} \quad \mathbb{C}^2 \times ALE_\Gamma \quad \stackrel{\mathrm{Id}\times qK}{\longleftarrow} \quad \mathbb{C}^2 \times \mathbb{V}_{|\Gamma|+1} \quad \stackrel{\mathscr{A}_{\mathscr{P}}}{\hookrightarrow} \quad \mathbb{C}^2 \times \mathbb{C}^{2|\Gamma|} \tag{3.35}$$

In the above equation the map $\stackrel{\mathfrak{H}^{-1}}{\longleftarrow}$ denotes the inverse of the harmonic function map on $\mathbb{C}^2 \times ALE_\Gamma$ that we have already discussed. The map $\stackrel{\mathrm{Id}\times qK}{\longleftarrow}$ is instead the product of the identity map $\mathrm{Id} : \mathbb{C}^2 \to \mathbb{C}^2$ with the Kähler quotient map:

$$qK \quad : \quad \mathbb{V}_{|\Gamma|+1} \longrightarrow \mathbb{V}_{|\Gamma|+1} /\!/_{K} \mathscr{F}_{|\Gamma|-1} \simeq ALE_\Gamma \tag{3.36}$$

of an algebraic variety of complex dimension $|\Gamma|+1$ with respect to a suitable Lie group $\mathscr{F}_{|\Gamma|-1}$ of real dimension $|\Gamma|-1$. Finally the map $\stackrel{\mathscr{A}_{\mathscr{P}}}{\hookrightarrow}$ denotes the inclusion map of the variety $\mathbb{V}_{|\Gamma|+1}$ in $\mathbb{C}^{2|\Gamma|}$. Let $y_1,\dots y_{2|\Gamma|}$ be the coordinates of $\mathbb{C}^{2|\Gamma|}$. The variety $\mathbb{V}_{|\Gamma|+1}$ is defined by an ideal generated by $|\Gamma|-1$ quadratic generators:

$$\begin{array}{rcl} \mathbb{V}_{|\Gamma|+1} & = & \mathbf{V}\left(\mathscr{I}_\Gamma\right) \\ \mathbb{C}\left[\mathbf{V}\left(\mathscr{I}_\Gamma\right)\right] & = & \frac{\mathbb{C}\left[y_1,\dots y_{2|\Gamma|}\right]}{\left(\mathscr{P}_1(y),\mathscr{P}_2(y),\dots,\mathscr{P}_{|\Gamma|-1}(y)\right)} \end{array} \tag{3.37}$$

Actually the $|\Gamma|-1$ polynomials $\mathscr{P}_i(y)$ are the holomorphic part of the triholomorphic moment maps associated with the triholomorphic action of the group $\mathscr{F}_{|\Gamma|-1}$ on $\mathbb{C}^{2|\Gamma|}$ and the entire procedure from $\mathbb{C}^{2|\Gamma|}$ to ALE_Γ can be seen as the hyperKähler quotient:

$$ALE_\Gamma = \mathbb{C}^{2|\Gamma|} /\!/_{HK} \mathscr{F}_{|\Gamma|-1} \tag{3.38}$$

yet we have preferred to split the procedure into two steps in order to compare case A) with case B) where the two steps are necessarily distinct and separated.

[4]The group L_{168} has three maximal subgroups, up to conjugation, namely two non conjugate copies of the octahedral group $\mathrm{O}_{24} \sim \mathrm{S}_4$ and one non abelian group of order 21, denoted G_{21} that is isomorphic to the semidirect product $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$.

Indeed in case B) we can write the following diagram:

$$\mathscr{M}_7 \quad \stackrel{\mathfrak{H}^{-1}}{\longleftarrow} \quad \mathbb{C}\times Y_\Gamma \quad \stackrel{\mathrm{Id}\times qK}{\longleftarrow} \quad \mathbb{C}\times\mathbb{V}_{|\Gamma|+2} \quad \stackrel{\mathrm{Id}\times\mathscr{A}_{\mathscr{P}}}{\hookrightarrow} \quad \mathbb{C}\times\mathbb{C}^{3|\Gamma|} \tag{3.39}$$

In this case, just as in the previous one, the intermediate step is provided by the Kähler quotient but the map on the extreme right $\stackrel{\mathscr{A}_{\mathscr{P}}}{\hookrightarrow}$ denotes the inclusion map of the variety $\mathbb{V}_{|\Gamma|+2}$ in $\mathbb{C}^{3|\Gamma|}$. Let $y_1,\dots y_{3|\Gamma|}$ be the coordinates of $\mathbb{C}^{3|\Gamma|}$. The variety $\mathbb{V}_{|\Gamma|+2}$ is defined as the principal branch of a set of quadratic algebraic equations that are group-theoretically defined. Altogether the mentioned construction singles out the holomorphic orbit of a certain group action to be discussed in detail in the sequel. So we anticipate:

$$\mathbb{V}_{|\Gamma|+2} \quad = \quad \mathscr{D}_\Gamma \equiv \mathrm{Orbit}_{\mathscr{G}_\Gamma}\left(L_\Gamma\right) \tag{3.40}$$

where both the set L_Γ and the complex group $\mathscr{G}_\Gamma$ are completely defined by the discrete group Γ defining the quotient singularity.

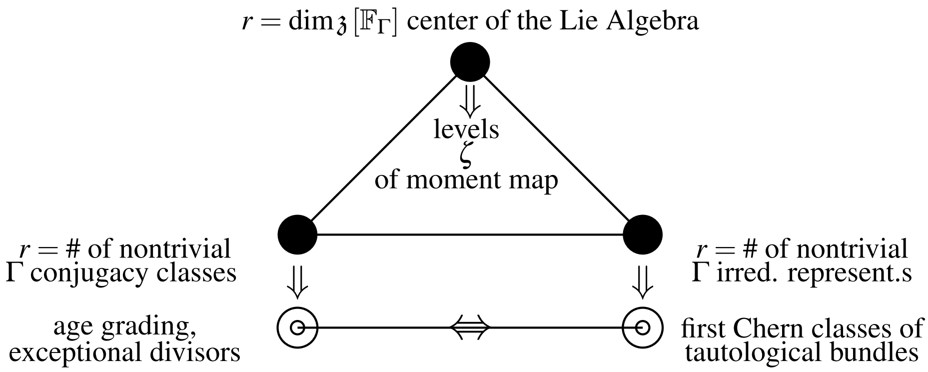
4 Generalities on $\frac{\mathbb{C}^3}{\Gamma}$ singularities

Recalling what we summarized above we conclude that the singularities relevant to our goals are of the form:

$$X = \frac{\mathbb{C}^3}{\Gamma} \tag{4.1}$$

where the finite group $\Gamma\subset\mathrm{SU}(3)$ has a holomorphic action on $\mathbb{C}^3$. For this case, as we mentioned above, there is a series of general results and procedures developed in algebraic geometry that we want to summarize in the perspective of their use in physics.

To begin with let us observe the schematic diagram sketched here below:



(4.2)

The fascination of the mathematical construction lying behind the desingularization process, which has a definite counterpart in the structure of the Chern-Simons gauge theories describing M2-branes at the $\mathbb{C}^3/\Gamma$ singularity, is the triple interpretation of the same number r which alternatively yields:

- The number of nontrivial conjugacy classes of the finite group Γ,
- The number of irreducible representations of the finite group Γ,

- The center of the Lie algebra $\mathfrak{z}[\mathbb{F}_\Gamma]$ of the compact gauge group $\mathscr{F}_\Gamma$, whose structure, as we will see, is:

$$\mathscr{F}_\Gamma = \bigotimes_{i=1}^{r} \mathrm{U(n_i)} \tag{4.3}$$

The levels ζ_I of the moment maps are the main ingredient of the singularity resolution. At level$\zeta^I = 0$ we have the singular orbifold $\mathscr{M}_0 = \frac{\mathbb{C}^3}{\Gamma}$, while at $\zeta^i \neq 0$ we obtain a smooth manifold $\mathscr{M}_\zeta$ which develops a nontrivial homology and cohomology. In physical parlance the levels ζ^I are the Fayet-Iliopoulos parameters appearing in the lagrangian, while $\mathscr{M}_\zeta$ is the manifold of vacua of the theory, namely of extrema of the potential, as we already emphasized.

Quite generally, we find that each of the gauge factors $\mathrm{U(n_i)}$ is the structural group of a holomorphic vector bundle of rank n_i:

$$\mathfrak{V}_i \xrightarrow{\pi} \mathscr{M}_\zeta \tag{4.4}$$

whose first Chern class is a nontrivial (1,1)-cohomology class of the resolved smooth manifold:

$$c_1(\mathfrak{V}_i) \in H^{1,1}\left(\mathscr{M}_\zeta\right) \tag{4.5}$$

On the other hand a very deep theorem originally proved in the nineties by Reid and Ito [36] relates the dimensions of the cohomology groups $H^{q,q}\left(\mathscr{M}_\zeta\right)$ to the conjugacy classes of Γ organized according to the grading named *age*. So named *junior classes* of *age* = 1 are associated with $H^{1,1}\left(\mathscr{M}_\zeta\right)$ elements, while the so-named *senior classes* of *age* = 2 are associated with $H^{2,2}\left(\mathscr{M}_\zeta\right)$ elements.

The link that pairs irreps with conjugacy classes is provided by the relation, well-known in algebraic geometry, between *divisors* and *line bundles*. The conjugacy classes of γ can be put into correspondence with the exceptional divisors created in the resolution $\mathscr{M}_\zeta \xrightarrow{\zeta\to 0} \frac{\mathbb{C}^3}{\Gamma}$ and each divisor defines a line bundle whose first Chern class is an element of the $H^{1,1}\left(\mathscr{M}_\zeta\right)$ cohomology group.

These line bundles labeled by conjugacy classes have to be compared with the line bundles created by the Kähler quotient procedure that are instead associated with the irreps, as we have sketched above. In this way we build the bridge between conjugacy classes and irreps.

Finally there is the question whether the divisor is compact or not. In the first case, by Poincaré duality, we obtain nontrivial $H^{2,2}\left(\mathscr{M}_\zeta\right)$ elements. In the second case we have no new cohomology classes. The age grading precisely informs us about the compact or noncompact nature of the divisors. Each senior class corresponds to a cohomology class of degree 4, thus signaling the existence of a non-trivial closed (2,2) form, and via Poincaré duality, it also corresponds to a compact component of the exceptional divisor.

The physics–friendly illustration of this general beautiful scheme, together with the explicit construction of a few concrete examples is the main goal of the present paper. We begin with the concept of age grading.

4.1 The concept of aging for conjugacy classes of the discrete group Γ

According to the above quoted theorem that we shall explain below, the *age grading* of Γ conjugacy classes allows to predict the Dolbeaults cohomology of the resolved algebraic variety. It goes as follows.

Suppose that Γ (a finite group) acts in a linear way on $\mathbb{C}^n$. Consider an element $\gamma \in \Gamma$ whose action is the following:

$$\gamma.\vec{z} = \underbrace{\begin{pmatrix} \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}}_{\mathscr{Q}(\gamma)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \tag{4.6}$$

Since in a finite group all elements have a finite order, there exists $r \in \mathbb{N}$, such that $\gamma^r = \mathbf{1}$. We define the age of an element in the following way. Let us diagonalize $D(\gamma)$, namely compute its eigenvalues. They will be as follows:

$$(\lambda_1,\ldots,\lambda_n) = \exp\left[\frac{2\pi i}{r} a_i\right] \quad ; \quad r > a_i \in \mathbb{N} \quad i = 1,\ldots,n \tag{4.7}$$

We define:

$$\text{age}\,(\gamma) = \frac{1}{r}\sum_{i=1}^{n} a_i \tag{4.8}$$

Clearly the age is a property of the conjugacy class of the element, relative to the considered three-dimensional complex representation.

4.2 The fundamental theorem

In [36] Y. Ito and M. Reid proved the following fundamental theorem:

Theorem 4.1 *Let $Y \to \mathbb{C}^3/\Gamma$ be a crepant* [5] *resolution of a Gorenstein* [6] *singularity. Then we have the following relation between the de-Rham cohomology groups of the resolved smooth variety Y and the ages of Γ conjugacy classes:*

$$\dim H^{2k}(Y) = \#\ \textit{of age } k \textit{ conjugacy classes of } \Gamma$$

On the other hand it happens that all odd cohomology groups are trivial:

$$\dim H^{2k+1}(Y) = 0 \tag{4.9}$$

This is true also in the case of $\mathbb{C}^2/\Gamma$ singularities, yet in $n = 2,3$ the consequences of the same fact are drastically different. In all complex dimensions n the deformations of the Kähler class are in one-to-one correspondence with the harmonic forms $\omega^{(1,1)}$, while those of the complex structure are in correspondence with the harmonic forms $\omega^{(n-1,1)}$. In $n = 2$ the harmonic $\omega^{(1,1)}$ forms play the double role of Kähler class deformations and complex structure deformations. This is the reason why we can do a hyperKähler quotient and we have both moduli parameters in the Kähler potential and in the polynomials cutting out the smooth variety. Instead in $n = 3$ eq. (4.9) implies that the polynomials constraints cutting the singular locus have no deformation parameters. The parameters of the resolution occur only at the level of the Kähler quotient and are the levels of the Kählerian moment maps.

Given an algebraic representation of the variety Y as the vanishing locus of certain polynomials $W(x) = 0$, the algebraic $2k$-cycles are the $2k$-cycles that can be holomorphically embedded in Y. The following statement in $n = 3$ is elementary:

Statement 4.1 *The Poincaré dual of any algebraic $2k$-cycle is of type (k,k).*

Its converse is known as the Hodge conjecture, stating that any cycle of type (k,k) is a linear combination of algebraic cycles. This will hold true for the varieties we shall be considering.

Thus we conclude that the so named *junior conjugacy classes* (age=1) are in a one-to-one correspondence with $\omega^{(1,1)}$-forms that span $H^{1,1}$, while *conjugacy classes of age 2* are in one-to-one correspondence with $\omega^{(2,2)}$-forms that span $H^{2,2}$.

[5] A resolution of singularities $X \to Y$ is crepant when the canonical bundle of X is the pullback of the canonical bundle of Y.

[6] A variety is Gorenstein when the canonical divisor is a Cartier divisor, i.e., a divisor corresponding to a line bundle.

5 Comparison with ALE manifolds and comments

Let us compare the above predictions for the case B) of $\mathbb{C}^3/\Gamma$ singularities with the well known case A) of $\mathbb{C}^2/\Gamma$ where Γ is a Kleinian subgroup of $\mathrm{SU}(2)$ and the resolution of the singularity leads to an ALE manifold [2, 30, 31, 38–40]. As we already stressed above, this latter can be explicitly constructed by means of a hyperKähler quotient, according to Kronheimer's construction.

In table 1 we summarize some well known facts about $Y \to X = \mathbb{C}^2/\Gamma$ which are the following. Here χ denotes Hirzebruch's signature characteristic of the resolved manifold.

Table 1: Finite $\mathrm{SU}(2)$ subgroups versus ALE manifold properties

Γ.	$W_\Gamma(u,w,z)$	$\mathscr{R} = \frac{\mathbf{C}[u,w,z]}{\partial W}$	$\lvert\mathscr{R}\rvert$	$\#c.\ c.$	$\tau \equiv \chi - 1$
A_k	$u^2+w^2-z^{k+1}$	$\{1,z,..$ $..,z^{k-1}\}$	k	$k+1$	k
D_{k+2}	$u^2+w^2z+z^{k+1}$	$\{1,w,z,w^2,$ $z^2,\dots,z^{k-1}\}$	$k+2$	$k+3$	$k+2$
$E_6 =$ $\mathscr{T}$	$u^2+w^3+z^4$	$\{1,w,z,$ $wz,z^2,wz^2\}$	6	7	6
$E_7 =$ $\mathscr{O}$	$u^2+w^3+wz^3$	$\{1,w,z,w^2,$ $z^2,wz,w^2z\}$	7	8	7
$E_8 =$ $\mathscr{I}$	$u^2+w^3+z^5$	$\{1,w,z,z^2,wz,$ $z^3,wz^2,wz^3\}$	8	9	8

1. As an affine variety the singular orbifold X is described by a single polynomial equation $W_\Gamma(u,w,z) = 0$ in $\mathbb{C}^3$. This equation is simply given by a relation existing among the invariants of Γ as we anticipated in the previous section. Note that this is the case also for $X = \frac{\mathbb{C}^3}{\mathrm{L}_{168}}$, as Markushevich has shown. He has found one polynomial constraint $W_{\mathrm{L}_{168}}(u_1,u_2,u_3,u_4) = 0$ of degree 10 in $\mathbb{C}^4$ which describes X. We were able to find a similar result for the subgroup $\mathrm{G}_{21} \subset \mathrm{L}_{168}$ and obviously also for the cases $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_7$. In the G_{21} case the equation is of order 16. We will present these results in a future publication.

2. The resolved locus Y in the case of ALE manifolds is described by a deformed equation:

$$\begin{aligned} W_\Gamma^{ALE}(u,w,z;\mathbf{t}) &= W_\Gamma(u,w,z) + \sum_{i=1}^{r} t_i\,\mathscr{P}^{(i)}(u,w,z) \\ r &\equiv \dim\mathscr{R}_\Gamma \end{aligned} \tag{5.1}$$

where

a) $W_\Gamma(u,w,z)$ is the simple singularity polynomial corresponding to the finite subgroup $\Gamma \subset \mathrm{SU}(2)$ according to Arnold's classification of isolated critical points of functions [41], named simple singularities in the literature.

b) $\mathscr{P}^{(i)}(u,w,z)$ is a basis spanning the chiral ring

$$\mathscr{R}_\Gamma = \frac{\mathbb{C}[u,w,z]}{\partial W_\Gamma} \tag{5.2}$$

of polynomials in u,w,z that do not vanish upon use of the vanishing relations $\partial_u W_\Gamma = \partial_w W_\Gamma = \partial_z W_\Gamma = 0$.

c) The complex parameters t^i are the complex structure moduli and they are in one–to–one correspondence with the set of complex level parameters $\ell_+^{\mathbf{X}}$.

3. According to the general view put forward in the previous section, for ALE manifolds we have:

$$\dim H^{1,1} = r \equiv \#\text{ nontrivial conjugacy classes of }\Gamma \tag{5.3}$$

We also have:

$$\dim\mathscr{R}_\Gamma = r \tag{5.4}$$

as one sees from table 1. From the point of view of complex geometry this is the consequence of a special coincidence, already stressed in the previous section, which applies only to the case of complex dimension 2. As one knows, for Calabi-Yau n-folds complex structure deformations are associated with $\omega^{n-1,1} \in H^{n-1,1}$ harmonic forms, while Kähler structure deformations, for all n, are associated with $\omega^{1,1} \in H^{1,1}$ harmonic forms. Hence when $n=2$, the $(1,1)$-forms play a double role as complex structure deformations and as Kähler structure deformations. For instance, this is well known in the case of $K3$. Hence in the $n=2$ case the number of *nontrivial conjugacy classes* of the group Γ coincides both with the number of Kähler moduli and with number of complex structure moduli of the resolved variety.

4. In the case of $Y \to X = \frac{\mathbb{C}^3}{\Gamma}$ the number of $(1,1)$-forms and hence of Kähler moduli is still related with $r = \#$ *junior conjugacy classes of* Γ but there are no complex structure deformations.

5.1 The McKay correspondence for $\mathbb{C}^2/\Gamma$

The table of characters $\chi_i^{(\mu)}$ of any finite group γ allows to reconstruct the decomposition coefficients of any representation along the irreducible representations:

$$\begin{aligned} D &= \bigoplus_{\mu=1}^{r} a_\mu D_\mu \\ a_\mu &= \frac{1}{g}\sum_{i=1}^{r} g_i\,\chi_i^{(D)}\,\chi_i^{(\mu)\star} \end{aligned} \tag{5.5}$$

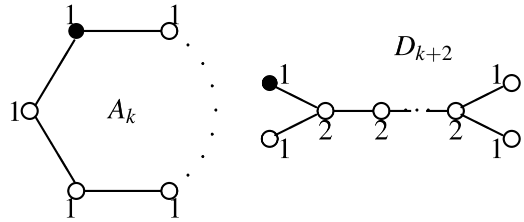


Figure 1: Extended Dynkin diagrams of the infinite series

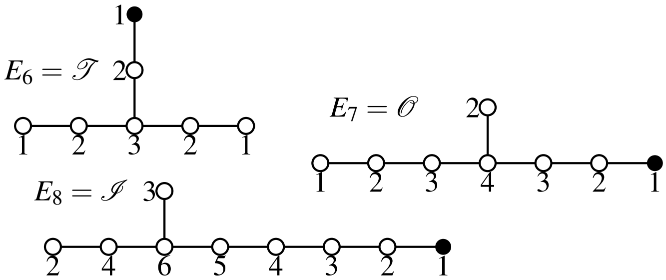


Figure 2: Exceptional extended Dynkin diagrams

where $\chi^{(D)}$ is the character of D. For the finite subgroups $\Gamma \subset \mathrm{SU}(2)$ a particularly important case is the decomposition of the tensor product of an irreducible representation D_μ with the defining 2-dimensional representation $\mathscr{Q}$. It is indeed at the level of this decomposition that the relation between these groups and the simply laced Dynkin diagrams becomes explicit and is named the McKay correspondence. This decomposition plays a crucial role in the explicit construction of ALE manifolds according to Kronheimer. Setting

$$\mathscr{Q} \otimes D_\mu \; = \; \bigoplus_{\nu=0}^{r} A_{\mu\nu} D_\nu \tag{5.6}$$

where D_0 denotes the identity representation, one finds that the matrix $\bar{c}_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}$ is the *extended Cartan matrix* relative to the *extended Dynkin diagram* corresponding to the given group. We remind the reader that the extended Dynkin diagram of any simply laced Lie algebra is obtained by adding to the *dots* representing the *simple roots* $\{\alpha_1 \ldots\ldots \alpha_r\}$ an *additional dot* (marked black in figs. 1, 2) representing the negative of the highest root $\alpha_0 = \sum_{i=1}^{r} n_i \alpha_i$ (n_i are the Coxeter numbers). Thus we see a correspondence between the nontrivial conjugacy classes $\mathscr{C}_i$ (or equivalently the nontrivial irreps) of the group $\Gamma(\mathbb{G})$ and the simple roots of $\mathbb{G}$. In this correspondence the extended Cartan matrix provides the Clebsch-Gordon coefficients (5.6), while the Coxeter numbers n_i express the dimensions of the irreducible representations. All these informations are summarized in Figs. 1, 2 where the numbers n_i are attached to each of the dots: the number 1 is attached to the extra dot since it stands for the identity representation.

5.2 Kronheimer's construction

Given any finite subgroup $\Gamma \subset \mathrm{SU}(2)$, we consider a space $\mathscr{P}$ whose elements are two-vectors of $|\Gamma| \times |\Gamma|$ complex matrices: $(A, B) \in \mathscr{P}$. The action of an element $\gamma \in \Gamma$ on the points of $\mathscr{P}$ is the following:

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ i v_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} R(\gamma) A R(\gamma^{-1}) \\ R(\gamma) B R(\gamma^{-1}) \end{pmatrix} \tag{5.7}$$

where the two-dimensional matrix on the right hand side is the realization of γ inside the defining two-dimensional representation $\mathcal{Q} \subset \mathrm{SU}(2)$, while $R(\gamma)$ is the regular, $|\Gamma|$-dimensional representation. The basis vectors in R named e_γ are in one-to-one correspondence with the group elements $\gamma \in \Gamma$ and transform as follows:

$$R(\gamma)\, e_\delta \;=\; e_{\gamma\cdot\delta} \quad \forall \gamma, \delta \in \Gamma \tag{5.8}$$

Intrinsically, the space $\mathscr{P}$ is named as:

$$\mathscr{P} \simeq \mathrm{Hom}\,(R, \mathcal{Q} \otimes R) \tag{5.9}$$

Next we introduce the space $\mathscr{S}$, which by definition is the subspace made of Γ-invariant elements in $\mathscr{P}$:

$$\mathscr{S} \equiv \{p \in \mathscr{P} / \forall \gamma \in \Gamma, \gamma \cdot p = p\} \tag{5.10}$$

Explicitly the invariance condition reads as follows:

$$\begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ i v_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} R(\gamma^{-1}) A R(\gamma) \\ R(\gamma^{-1}) B R(\gamma) \end{pmatrix} \tag{5.11}$$

The decomposition (5.6) is very useful in order to determine the Γ-invariant vector space (5.10).

A two-vector of matrices can be thought of also as a matrix of two-vectors: that is, $\mathscr{P} = \mathcal{Q} \otimes \mathrm{Hom}(R,R) = \mathrm{Hom}(R, \mathcal{Q} \otimes R)$. Decomposing the regular representation, $R = \bigoplus_{\nu=0}^{r} n_\mu D_\mu$ into irreps, using eq. (5.6) and Schur's lemma, we obtain:

$$\mathscr{S} = \bigoplus_{\mu,\nu} A_{\mu,\nu} \mathrm{Hom}(\mathbb{C}^{n_\mu}, \mathbb{C}^{n_\nu}) \,. \tag{5.12}$$

The dimensions of the irreps, n_μ are dispayed in figs. 1,2. From eq. (5.12) the real dimension of $\mathscr{S}$ follows immediately: $\dim \mathscr{S} = \sum_{\mu,\nu} 2A_{\mu\nu} n_\mu n_\nu$ implies, recalling that $A = 2 \times \mathbf{1} - \bar{c}$ [see eq. (5.6)] and that for the extended Cartan matrix $\bar{c}n = 0$:

$$\dim_{\mathbb{C}} \mathscr{S} = 2 \sum_\mu n_\mu^2 = 2|\Gamma| \,. \tag{5.13}$$

Intrinsically, one writes the space $\mathscr{S}$ as:

$$\mathscr{S} \simeq \mathrm{Hom}_\Gamma\,(R, \mathcal{Q} \otimes R) \tag{5.14}$$

So we can summarize the discussion by saying that:

$$\dim_{\mathbb{C}} \left[\mathrm{Hom}_\Gamma\,(R, \mathcal{Q} \otimes R)\right] = 2\,|\Gamma| \tag{5.15}$$

The quaternionic structure of the flat manifolds $\mathscr{P}$ and $\mathscr{S}$ can be seen by simply writing their elements as follows:

$$p = \begin{pmatrix} A & iB^\dagger \\ iB & A^\dagger \end{pmatrix} \in \mathrm{Hom}\,(R, \mathcal{Q} \otimes R) \qquad A, B \in \mathrm{End}(R) \,.$$

Then the hyperKähler forms and the hyperKähler metric are defined by the following formulae:

$$\begin{aligned} \Theta &= \mathrm{Tr}(\mathrm{d}p^\dagger \wedge \mathrm{d}p) = \begin{pmatrix} \mathrm{i}\mathbf{K} & \mathrm{i}\overline{\mathbf{\Omega}} \\ \mathrm{i}\mathbf{\Omega} & -\mathrm{i}\mathbf{K} \end{pmatrix} \\ ds^2 \times \mathbf{1} &= \mathrm{Tr}(\mathrm{d}p^\dagger \otimes \mathrm{d}p) \end{aligned} \tag{5.16}$$

In the above equations the trace is taken over the matrices belonging to $\mathrm{End}(R)$ in each entry of the quaternion.

From eq. (5.16) we extract the explicit expressions for the Kähler 2-form $\mathbf{K}$ and the holomorphic 2-form $\mathbf{\Omega}$ of the flat hyperKähler manifold $\mathrm{Hom}\,(R, \mathscr{Q} \otimes R)$. We have:

$$\begin{aligned}
\mathbf{K} &= -\mathrm{i}\left[\mathrm{Tr}\left(\mathrm{d}A^\dagger \wedge \mathrm{d}A\right) + \mathrm{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}B\right)\right] \equiv \mathrm{i} g_{\alpha\overline{\beta}}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^{\overline{\beta}} \\
ds^2 &= g_{\alpha\overline{\beta}}\, \mathrm{d}q^\alpha \otimes \mathrm{d}q^{\overline{\beta}} \\
\mathbf{\Omega} &= 2\mathrm{Tr}\left(\mathrm{d}A \wedge \mathrm{d}B\right) \equiv \Omega_{\alpha\beta}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta
\end{aligned} \tag{5.17}$$

Starting from the above written formulae, by means of an elementary calculation one verifies that both the metric and the hyperKähler forms are invariant with respect to the action of the discrete group Γ defined in eq. (5.7). Hence one can consistently reduce the space $\mathrm{Hom}\,(R, \mathscr{Q} \otimes R)$ to the invariant space $\mathrm{Hom}_\Gamma\,(R, \mathscr{Q} \otimes R)$ defined in eq. (5.10). The hyperKähler 2-forms and the metric of the flat space $\mathscr{S}$, whose real dimension is $4|\Gamma|$, are given by eqs.(5.17) where the matrices A, B satisfy the invariance condition eq. (5.11).

5.2.1 Solution of the invariance constraint in the case of the cyclic groups A_k

The space $\mathscr{S}$ can be easily described when Γ is the cyclic group $A_k = \mathbb{Z}_{k+1}$, whose multiplication table can be read off. We can immediately read it off from the matrices of the regular representation. Obviously, it is sufficient to consider the representative of the first element e_1, as $R(e_j) = (R(e_1))^j$.

One has:

$$R(e_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{5.18}$$

Actually, the invariance condition eq. (5.11) is best solved by changing basis so as to diagonalize the regular representation, realizing explicitly its decomposition in terms of the k unidimensional irreps. Let $\nu = e^{\frac{2\pi i}{k+1}}$, be a $(k+1)$th root of unity so that $\nu^{k+1} = 1$. The looked for change of basis is performed by means of the matrix:

$$\begin{aligned}
S_{ij} &= \frac{1}{\sqrt{k+1}} \nu^{ij} \quad ; \quad i, j = 0, 1, 2, \ldots, k \\
\left(S^{-1}\right)_{ij} = \left(S^\dagger\right)_{ij} &= \frac{1}{\sqrt{k+1}} \nu^{k+1-ij}
\end{aligned} \tag{5.19}$$

In the new basis we find:

$$
\begin{aligned}
\widehat{R}(e_0) &\equiv S^{-1}R(e_0)S = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0\dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \\
\widehat{R}(e_1) &\equiv S^{-1}R(e_1)S = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \nu & 0\dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \nu^{k-1} & 0 \\ 0 & 0 & \dots & 0 & \nu^k \end{pmatrix}
\end{aligned} \tag{5.20}
$$

Eq. (5.20) displays on the diagonal the representatives of e_j in the one-dimensional irreps.

In the above basis, the explicit solution of eq. (5.11) is given by

$$
A = \begin{pmatrix} 0 & u_0 & 0 & \cdots & 0 \\ 0 & 0 & u_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & u_{k-1} \\ u_k & 0 & 0 & \cdots & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & 0 & \cdots & \cdots & v_k \\ v_0 & 0 & \cdots & \cdots & 0 \\ 0 & v_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & v_{k-1} & 0 \end{pmatrix} \tag{5.21}
$$

We see that these matrices are parameterized in terms of $2k+2$ complex, i.e. $4(k+1) = |A_k|$ real parameters. In the D_{k+2} case, where the regular representation is $4k$-dimensional, choosing appropriately a basis, one can solve analogously eq. (5.11); the explicit expressions are too large, so we do not write them. The essential point is that the matrices A and B no longer correspond to two distinct set of parameters, the group being nonabelian.

5.3 The gauge group for the quotient and its moment maps

The next step in the Kronheimer construction of the ALE manifolds is the determination of the group $\mathscr{F}$ of triholomorphic isometries with respect to which we will perform the quotient. We borrow from physics the nomenclature *gauge group* since in a $\mathscr{N}=3,4$ rigid three-dimensional gauge theory where the space $\mathrm{Hom}_\Gamma(R, \mathscr{Q}\otimes R)$ is the flat manifold of hypermultiplet scalars, the triholomorphic moment maps of $\mathscr{F}$ emerge as scalar dependent nonderivative terms in the hyperino supersymmetry transformation rules generated by the *gauging* of the group $\mathscr{F}$.

Consider the action of $\mathrm{SU}(|\Gamma|)$ on $\mathrm{Hom}(R, \mathscr{Q}\otimes R)$ given, using the quaternionic notation for the elements of $\mathrm{Hom}(R, \mathscr{Q}\otimes R)$, by

$$
\forall g \in \mathrm{SU}(|\Gamma|) \quad , \quad g \quad : \quad \begin{pmatrix} A & iB^\dagger \\ iB & A^\dagger \end{pmatrix} \longmapsto \begin{pmatrix} gAg^{-1} & igB^\dagger g^{-1} \\ igBg^{-1} & gA^\dagger g^{-1} \end{pmatrix} \tag{5.22}
$$

It is easy to see that this action is a triholomorphic isometry of $\mathrm{Hom}(R, \mathscr{Q}\otimes R)$. Indeed both the hyperKähler forms Θ and the metric ds^2 are invariant.

Let $\mathscr{F} \subset \mathrm{SU}(|\Gamma|)$ be the subgroup of the above group which *commutes with the action of* Γ *on the space* $\mathrm{Hom}\,(R,\mathscr{Q}\otimes R)$, action which was defined in eq. (5.7). Then the action of $\mathscr{F}$ descends to $\mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R) \subset \mathrm{Hom}\,(R,\mathscr{Q}\otimes R)$ to give a *triholomorphic isometry*: indeed the metric and the hyperKähler forms on the space $\mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R)$ are just the restriction of those on $\mathrm{Hom}\,(R,\mathscr{Q}\otimes R)$. Therefore one can take the hyperKähler quotient of $\mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R)$ with respect to $\mathscr{F}$.

Let $\{f_A\}$ be a basis of generators for $\mathbb{F}$, the Lie algebra of $\mathscr{F}$. Under the infinitesimal action of $f = \mathbf{1} + \lambda^A f_A \in \mathbb{F}$, the variation of $p \in \mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R)$ is $\delta p = \lambda^A \delta_A p$, with

$$\delta_A p = \begin{pmatrix} [f_A,A] & i[f_A,B^\dagger] \\ i[f_A,B] & [f_A,A^\dagger] \end{pmatrix}$$

The components of the momentum map are then given by

$$\mu_A = \mathrm{Tr}\,(q^\dagger\, \delta_A p) \;\equiv\; \mathrm{Tr}\begin{pmatrix} f_A\,\mu_3(p) & f_A\,\mu_-(p) \\ f_A\,\mu_+(p) & f_A\,\mu_3(p) \end{pmatrix}$$

so that the real and holomorphic maps $\mu_3 : \mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R) \to \mathbb{F}^*$ and $\mu_+ : \mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R) \to \mathbb{C}\times\mathbb{F}^*$ can be represented as matrix-valued maps:

$$\begin{aligned} \mu_3(p) &= -i\left([A,A^\dagger]+[B,B^\dagger]\right) \\ \mu_+(p) &= ([A,B])\ . \end{aligned} \tag{5.23}$$

In this way we get:

$$\mu_A = \begin{pmatrix} \mathfrak{P}^3_A & \mathfrak{P}^-_A \\ \mathfrak{P}^+_A & -\mathfrak{P}^3_A \end{pmatrix} \tag{5.24}$$

where:

$$\begin{aligned} \mathfrak{P}^3_A &= -\mathrm{i}\left[\mathrm{Tr}\left(\left[A\,,A^\dagger\right] f_A\right) + \mathrm{Tr}\left(\left[B^\dagger\,,B\right] f_A\right)\right] \\ \mathfrak{P}^+_A &= \mathrm{Tr}\left([A\,,B]\, f_A\right) \end{aligned} \tag{5.25}$$

Let $\mathfrak{Z}^\star$ be the dual of the center of $\mathbb{F}$.

In correspondence with a level $\zeta = \{\zeta^3,\zeta^+\} \in \mathbb{R}^3\otimes\mathfrak{Z}^\star$ we can form the hyperKähler quotient:

$$\mathscr{M}_\zeta \equiv \mu^{-1}(\zeta)\,/\!/_{_{HK}}\,\mathscr{F} \tag{5.26}$$

Varying ζ and Γ all ALE manifolds can be obtained as $\mathscr{M}_\zeta$.

First of all, it is not difficult to check that $\mathscr{M}_\zeta$ is four-dimensional. Let us see how this happens. There is a nice characterization of the group $\mathscr{F}$ in terms of the extended Dynkin diagram associated with Γ. We have

$$\mathscr{F} = \bigotimes_{\mu=1}^{r+1} \mathrm{U}(\mathrm{n}_\mu) \bigcap \mathrm{SU}(|\Gamma|) \tag{5.27}$$

where the sum is extended to all the irreducible representations of the group Γ and n_μ are their dimensions. One should also take into account that the determinant of all the elements must be one, since $\mathscr{F} \subset \mathrm{SU}(|\Gamma|)$. Pictorially the group $\mathscr{F}$ has a $\mathrm{U}(\mathrm{n}_\mu)$ factor for each dot of the diagram, n_μ being associated with the dots as in figs. 1,2. $\mathscr{F}$ acts on the various *components* of $\mathrm{Hom}_\Gamma\,(R,\mathscr{Q}\otimes R)$ that are in correspondence with the edges of the diagram, see eq. (5.12), as dictated by the diagram structure. From eq. (5.27) it is immediate to derive:

$$\dim \mathscr{F} = \sum_{\mu} n_{\mu}^2 - 1 = |\Gamma| - 1 \tag{5.28}$$

It follows that

$$\dim_{\mathbb{R}} \mathscr{M}_{\zeta} = \dim_{\mathbb{R}} \mathrm{Hom}_{\Gamma}(R, \mathscr{Q} \otimes R) - 4 \dim_{\mathbb{R}} \mathscr{F} = 4|\Gamma| - (4|\Gamma| - 1) = 4 \tag{5.29}$$

Analyzing the construction we see that there are two steps. In the first step, by setting the holomorphic part of the moment map to its level ζ, we define an algebraic locus in $\mathrm{Hom}_{\Gamma}(\mathscr{Q} \otimes R, R)$. Next the Kähler quotient further reduces such a locus to the necessary complex dimension 2. The two steps are united in one because of the triholomorphic character of the isometries. As we already stressed in the previous section, in complex dimension 3 the isometries are not triholomorphic rather just holomorphic; hence the holomorphic part of the moment map does not exist and the two steps have to be separated. There must be another principle that leads to impose those constraints that cut out the algebraic locus $\mathbb{V}_{|\Gamma|+2}$ of which we perform the Kähler quotient in the next step (see eq. (3.40)). The main question is to spell out such principles. As anticipated, equation $\mathbf{p} \wedge \mathbf{p} = 0$ is the one that does the job. We are not able to reduce the $3|\Gamma|^2$ quadrics on $3|\Gamma|$ variables to an ideal with $2|\Gamma| - 2$ generators, yet we know that such reduction must exist. Indeed, by means of another argument that utilizes Lie group orbits we can show that there is a variety of complex dimension 3, named $\mathscr{D}_{\Gamma}^{0}$ which is in the kernel of the equation $\mathbf{p} \wedge \mathbf{p} = 0$.

5.3.1 The triholomorphic moment maps in the A_k case of Kronheimer construction

The structure of $\mathscr{F}$ and the momentum map for its action are very simply worked out in the A_k case. An element $f \in \mathscr{F}$ must commute with the action of A_k on $\mathscr{P}$, eq. (5.7), where the two-dimensional representation in the l.h.s. is given by:

$$\Gamma(A_k)\,\gamma_{\ell} = \mathscr{Q}_{\ell} \equiv \begin{pmatrix} e^{2\pi i \ell/(k+1)} & 0 \\ 0 & e^{-2\pi i \ell/(k+1)} \end{pmatrix} \quad ; \quad \{\ell = 1, \dots\dots, k+1\}$$

Then f must have the form

$$f = \mathrm{diag}(e^{i\varphi_0}, e^{i\varphi_1}, \dots, e^{i\varphi_k}) \;\; ; \;\; \sum_{i=0}^{k} \varphi_i = 0\,. \tag{5.30}$$

Thus $\mathbb{F}$ is just the algebra of diagonal traceless $k+1$-dimensional matrices, which is k-dimensional. Choose a basis of generators for $\mathbb{F}$, for instance:

$$\begin{aligned} f_1 &= \mathrm{diag}(1, -1, 0, \dots, 0) \\ f_2 &= \mathrm{diag}(1, 0, -1, 0, \dots, 0) \\ \dots &= \dots \\ f_k &= \mathrm{diag}(1, 0, 0, \dots, 0, -1) \end{aligned} \tag{5.31}$$

From eq. (5.25) we immediately obtain the components of the momentum map:

$$\begin{aligned} \mathfrak{P}_A^3 &= |u^0|^2 - |u^k|^2 - |v_0|^2 + |v_k|^2 + \left(|u^{A-1}|^2 - |u^A|^2 - |v_{A-1}|^2 + |v_A|^2\right) \\ \mathfrak{P}_A^+ &= u^0 v_0 - u^k v_k + \left(u^{A-1} v_{A-1} - u^A v_A\right) \quad , \quad (A = 1, \dots, k) \end{aligned} \tag{5.32}$$

5.4 Level sets and Weyl chambers

If $\mathscr{F}$ acts freely on $\mu^{-1}(\zeta)$ then $\mathscr{M}_\zeta$ is a smooth manifold. This happens or does not happen depending on the value of ζ. A simple characterization of $\mathfrak{Z}$ can be given in terms of the simple Lie algebra $\mathbb{G}$ associated with Γ. There exists an isomorphism between $\mathfrak{Z}$ and the Cartan subalgebra $\mathscr{H}_{CSA} \subset \mathbb{G}$. Thus we have

$$\begin{aligned} \dim \mathfrak{Z} = \dim \mathscr{H}_{CSA} &= \operatorname{rank} \mathbb{G} \\ &= \#\text{of nontrivial conj. classes in } \Gamma \end{aligned} \tag{5.33}$$

The space $\mathscr{M}_\zeta$ turns out to be singular when, under the above identification $\mathfrak{Z} \sim \mathscr{H}_{CSA}$, any of the level components $\zeta^i \in \mathbb{R}^3 \otimes \mathfrak{Z}$ lies on a wall of a Weyl chamber. In particular, as the point $\zeta^i = 0$, $(i = 1, \ldots, r)$ is identified with the origin of the root space, which lies of course on all the walls of the Weyl chambers, *the space $\mathscr{M}_0$ is singular*. Not too surprisingly we will see in a moment that $\mathscr{M}_0$ corresponds to the *orbifold limit* $\mathbb{C}^2/\Gamma$ of a family of ALE manifolds with boundary at infinity $\mathbb{S}^3/\Gamma$.

To verify this statement in general let us choose the natural basis for the regular representation R, in which the basis vectors e_δ transform as in eq. (5.8). Define the space $L \subset \mathscr{S}$ as follows:

$$L = \left\{ \begin{pmatrix} C \\ D \end{pmatrix} \in \mathscr{S} / C, D \text{ are diagonal in the basis } \{e_\delta\} \right\} \tag{5.34}$$

For every element $\gamma \in \Gamma$ there is a pair of numbers (c_γ, d_γ) given by the corresponding entries of C, D: $C \cdot e_\gamma = c_\gamma e_\gamma$, $D \cdot e_\gamma = d_\gamma e_\gamma$. Applying the invariance condition eq. (5.11), which is valid since $L \subset \mathscr{S}$, we obtain:

$$\begin{pmatrix} c_{\gamma\cdot\delta} \\ d_{\gamma\cdot\delta} \end{pmatrix} = \begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ iv_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} c_\delta \\ d_\delta \end{pmatrix} \tag{5.35}$$

We can identify L with $\mathbb{C}^2$ associating for instance $(C, D) \in L \longmapsto (c_0, d_0) \in \mathbb{C}^2$. Indeed all the other pairs (c_γ, d_γ) are determined in terms of eq. (5.35) once (c_0, d_0) are given. By eq. (5.35) the action of Γ on L induces exactly the action of Γ on $\mathbb{C}^2$ provided by its two-dimensional defining representation inside SU(2). It is quite easy to show the following fundamental fact: *each orbit of $\mathscr{F}$ in $\mu^{-1}(0)$ meets L in one orbit of* Γ. Because of the above identification between L and $\mathbb{C}^2$, this leads to conclude that $\mu^{-1}(0)/\mathscr{F}$ *is isometric to* $\mathbb{C}^2/\Gamma$. Instead of reviewing the formal proofs of these statements as devised by Kronheimer, we will verify them explicitly in the case of the cyclic groups, giving a description which sheds some light on the *deformed* situation; that is we show in which way a nonzero level ζ^+ for the holomorphic momentum map puts $\mu^{-1}(\zeta)$ in correspondence with the affine hypersurface in $\mathbb{C}^3$ cut out by the polynomial constraint (5.1) which is a deformation of that describing the $\mathbb{C}^2/\Gamma$ orbifold, obtained for $\zeta^+ = 0$.

5.4.1 Retrieving the polynomial constraint from the hyperKähler quotient in the Γ=A_k case.

We can directly realize $\mathbb{C}^2/\Gamma$ as an affine algebraic surface in $\mathbb{C}^3$ by expressing the coordinates x, y and z of $\mathbb{C}^3$ in terms of the matrices $(C, D) \in L$. The explicit parametrization of the matrices in $\mathscr{S}$ in the A_k case, which was given in eq. (5.21) in the basis in which the regular representation R is diagonal, can be conveniently rewritten in the *natural basis* $\{e_\gamma\}$ via the matrix S^{-1} defined in eq. (5.19). The subset L of diagonal matrices (C, D) is given by:

$$C = c_0 \operatorname{diag}(1, \nu, \nu^2, \ldots, \nu^k), \quad D = d_0 \operatorname{diag}(1, \nu^k, \nu^{k-1}, \ldots, \nu), \tag{5.36}$$

This is nothing but the fact that $L \sim \mathbb{C}^2$. The set of pairs $\begin{pmatrix} \nu^m c_0 \\ \nu^{k-m} d_0 \end{pmatrix}$, $m = 0, 1, \ldots, k$ is an orbit of Γ in $\mathbb{C}^2$ and determines the corresponding orbit of Γ in L. To describe $\mathbb{C}^2/A_k$ one needs to identify a suitable set of invariants $(u, w, z) \in \mathbb{C}^3$ such that

$$0 = W_\Gamma(u, w, z) \equiv u^2 + w^2 - z^{k+1} \tag{5.37}$$

To this effect we define:

$$u = \tfrac{1}{2}(x+y) \quad ; \quad w = -\mathrm{i}\tfrac{1}{2}(x-y) \quad \Rightarrow \quad xy = u^2 + w^2 \tag{5.38}$$

and we make the following ansatz:

$$x = \det C \quad ; \quad y = \det D, \quad ; \quad z = \frac{1}{k+1} \mathrm{Tr} CD. \tag{5.39}$$

This guess is immediately confirmed by the study of the deformed surface. We know that there is a one-to-one correspondence between the orbits of $\mathscr{F}$ in $\mu^{-1}(0)$ and those of Γ in L. Let us realize this correspondence explicitly.

Choose the basis where R is diagonal. Then $(A, B) \in \mathscr{S}$ have the form of eq. (5.21). The relation $xy = z^{k+1}$ holds also true when, in eq. (5.39), the pair $(C, D) \in L$ is replaced by an element $(A, B) \in \mu^{-1}(0)$.

To see this, let us describe the elements $(A, B) \in \mu^{-1}(0)$. We have to equate the right hand sides of eq. (5.23) to zero. We note that:

$$[A, B] = 0 \quad \Rightarrow \quad v_i = \frac{u_0 v_0}{u_i} \quad \forall i \tag{5.40}$$

Secondly,

$$[A, A^\dagger] + [B, B^\dagger] = 0 \quad \Rightarrow \quad |u_i| = |u_j| \,\text{and}\, |v_i| = |v_j| \quad \forall i, j \tag{5.41}$$

From the previous two equations we conclude that:

$$u_j = |u_0| \mathrm{e}^{i\phi_j} \quad ; \quad v_j = |v_0| \mathrm{e}^{i\psi_j} \tag{5.42}$$

Finally:

$$[A, B] = 0 \quad \Rightarrow \quad \psi_j = \Phi - \phi_j \quad \forall j \tag{5.43}$$

where Φ is an arbitrary overall phase.

In this way, we have characterized $\mu^{-1}(0)$ and we immediately check that the pair $(A, B) \in \mu^{-1}(0)$ satisfies $xy = z^{k+1}$ if $x = \det A$, $y = \det B$ and $z = 1/(k+1) \mathrm{Tr} AB$ as we have proposed in eq. (5.39).

After this explicit solution of the momentum map constraint has been implemented we are left with $k+4$ parameters, namely the $k+1$ phases ϕ_j, $j = 0, 1, \ldots k$, plus the absolute values $|u_0|$ and $|v_0|$ and the overall phase Φ. So we have:

$$\dim \mu^{-1}(0) = \dim \mathscr{S} - 3 \dim \mathscr{F} = 4|\Gamma| - 3(|\Gamma| - 1) = |\Gamma| + 3 \tag{5.44}$$

where $|\Gamma| = k+1$.

Now we perform the quotient of $\mu^{-1}(0)$ with respect to $\mathscr{F}$. Given a set of phases f_i such that $\sum_{i=0}^k f_i = 0 \bmod 2\pi$ and given $f = \mathrm{diag}(\mathrm{e}^{if_0}, \mathrm{e}^{if_1}, \ldots, \mathrm{e}^{if_k}) \in \mathscr{F}$, the orbit of $\mathscr{F}$ in $\mu^{-1}(0)$ passing through $\begin{pmatrix} A \\ B \end{pmatrix}$ has the form $\begin{pmatrix} fAf^{-1} \\ fBf^{-1} \end{pmatrix}$.

Choosing $f_j = f_0 + j\psi + \sum_{n=0}^{j-1} \phi_n$, $j = 1, \ldots, k$, with $\psi = -\frac{1}{k} \sum_{n=0}^{k} \phi_n$, and f_0 determined by the condition

$\sum_{i=0}^{k} f_i = 0 \bmod 2\pi$, one obtains

$$fAf^{-1} = a_0 \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad fBf^{-1} = b_0 \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & \dots & & \dots & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{5.45}$$

where $a_0 = |u_0|e^{i\psi}$ and $b_0 = |v_0|e^{i(\Phi-\psi)}$. Since the phases ϕ_j are determined modulo 2π, it follows that ψ is determined modulo $\frac{2\pi}{k+1}$. Thus we can say $(a_0, b_0) \in \mathbb{C}^2/\Gamma$. This is the one-to-one correspondence between $\mu^{-1}(0)/\mathscr{F}$ and $\mathbb{C}^2/\Gamma$.

Next we derive the deformed relation between the invariants x, y, z. It fixes the correspondence between the resolution of the singularity performed in the momentum map approach and the resolution performed on the hypersurface $xy = z^{k+1}$ in $\mathbb{C}^3$. To this purpose, we focus on the holomorphic part of the momentum map, i.e. on the equation:

$$[A,B] = \Lambda_0 = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_k) \in \mathfrak{Z} \otimes \mathbb{C} \tag{5.46}$$

$$\lambda_0 = -\sum_{i=1}^{k} \lambda_i \tag{5.47}$$

Let us recall the expression (5.21) for the matrices A and B. Naming $a_i = u_i v_i$, eq. (5.46) implies:

$$a_i = a_0 + \lambda_i \quad ; \quad i = 1, \dots, k \tag{5.48}$$

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$. We have

$$xy = \det A \, \det B = a_0 \Pi_{i=1}^{k} (a_0 + \lambda_i) = a_0^{k+1} \det\left(1 + \frac{1}{a_0}\Lambda\right) = \sum_{i=0}^{k} a_0^{k+1-i} S_i(\Lambda) \tag{5.49}$$

The $S_i(\Lambda)$ are the symmetric polynomials in the eigenvalues of Λ. They are defined by the relation $\det(1 + t\Lambda) = \sum_{i=0}^{k} t^i S_i(\Lambda)$ and are given by:

$$S_i(\Lambda) = \sum_{j_1 < j_2 < \dots < j_i} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i} \tag{5.50}$$

In particular, $S_0 = 1$ and $S_1 = \sum_{i=1}^{k} \lambda_i$. Define $S_{k+1}(\Lambda) = 0$, so that we can rewrite:

$$xy = \sum_{i=0}^{k+1} a_0^{k+1-i} S_i(\Lambda) \tag{5.51}$$

and note that

$$z = \frac{1}{k+1} \mathrm{Tr} AB = a_0 + \frac{1}{k+1} S_1(\Lambda). \tag{5.52}$$

Then the desired deformed relation between x, y and z is obtained by substituting $a_0 = z - \frac{1}{k} S_1$ in (5.49), thus obtaining

$$
\begin{aligned}
xy &= \sum_{m=0}^{k+1}\sum_{n=0}^{k+1-m}\begin{pmatrix} k+1-m \\ n \end{pmatrix} z^n \left(-\frac{1}{k+1}S_1\right)^{k+1-m-n} S_m z^n = \sum_{n=0}^{k+1} t_{n+1} z^n \\
\Longrightarrow \quad t_{n+1} &= \sum_{m=0}^{k+1-n}\begin{pmatrix} k+1-m \\ n \end{pmatrix}\left(-\frac{1}{k+1}S_1\right)^{k+1-m-n}
\end{aligned}
\tag{5.53}
$$

Note in particular that $t_{k+2} = 1$ and $t_{k+1} = 0$, i.e.

$$
xy = z^{k+1} + \sum_{n=0}^{k} t_{n+1} z^n \tag{5.54}
$$

which means that the deformation proportional to z^k is absent. This establishes a clear correspondence between the momentum map construction and the polynomial ring $\frac{\mathbb{C}[x,y,z]}{\partial W}$ where $W(x,y,z) = xy - z^{k+1}$. Moreover, note that we have only used one of the momentum map equations, namely $[A,B] = \Lambda_0$. The equation $[A,A^\dagger] + [B,B^\dagger] = \Sigma$ has been completely ignored. This means that the deformation of the complex structure is described by the parameters Λ, while the parameters Σ describe the deformation of the Kähler structure. The relation (5.53) can also be written in a simple factorized form, namely

$$
xy = \Pi_{i=0}^{k}(z - \mu_i), \tag{5.55}
$$

where

$$
\begin{aligned}
\mu_i &= \frac{1}{k}(\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} - 2\lambda_i + \lambda_{i+1} + \cdots + \lambda_k), \;\; i = 1, \ldots, k-1 \\
\mu_0 &= -\sum_{i=1}^{k} \mu_i = \frac{1}{k} S_1.
\end{aligned}
\tag{5.56}
$$

6 Generalization of the correspondence: McKay quivers for $\mathbb{C}^3/\Gamma$ singularities

One can generalize the extended Dynkin diagrams obtained in the above way by constructing McKay quivers, according to the following definition:

Definition 6.1 *Let us consider the quotient $\mathbb{C}^n/\Gamma$, where Γ is a finite group that acts on $\mathbb{C}^n$ by means of the complex representation $\mathcal{Q}$ of dimension n and let D_i, $(i = 1, \ldots, r+1)$ be the set of irreducible representations of Γ having denoted by $r+1$ the number of conjugacy classes of Γ. Let the matrix $\mathscr{A}_{ij}$ be defined by:*

$$
\mathcal{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{r+1} \mathscr{A}_{ij} \mathrm{D}_j \tag{6.1}
$$

To such a matrix we associate a quiver diagram in the following way. Every irreducible representation is denoted by a circle labeled with a number equal to the dimension of the corresponding irrep. Next we write an oriented line going from circle i to circle j if D_j appears in the decomposition of $\mathcal{Q} \otimes \mathrm{D}_i$, namely if the matrix element $\mathscr{A}_{ij}$ does not vanish.

The analogue of the extended Cartan matrix discussed in the case of $\mathbb{C}^2/\Gamma$ is defined below:

$$
\overline{c}_{ij} = n\,\delta_{ij} - \mathscr{A}_{ij} \tag{6.2}
$$

and it has the same property, namely, it admits the vector of irrep dimensions

$$\mathbf{n} \equiv \{1, n_1, \dots, n_r\} \tag{6.3}$$

as a null vector:

$$\bar{c}.\mathbf{n} = \mathbf{0} \tag{6.4}$$

Typically the McKay quivers encode the information determining the interaction structure of the dual gauge theory on the brane world volume. Indeed the bridge between Mathematics and Physics is located precisely at this point. In the case of a single $M2$-brane, the $n|\Gamma|$ complex coordinates (n=2, or 3) of the flat Kähler manifold $Hom_\Gamma(R, Q\otimes R)$ are the scalar fields of the Wess-Zumino multiplets, the unitary group $\mathscr{F}$ commuting with the action of Γ is the *gauge group*, the moment maps of $\mathscr{F}$ enter the definition of the potential, according to the standard supersymmetry formulae, recalled in section 2 and the holomorphic constraints defining the $\mathbf{V}_{|\Gamma|+2}$ variety have to be related with the superpotential $\mathfrak{W}$ of the $\mathscr{N}=2$ theories in d=3 (*i.e.* the n=3 case where the singular space is $\mathbb{C}\times\mathbb{C}^3/\Gamma$). In the case of $\mathscr{N}=4$ theories, also in d=3, (*i.e.* the n=2 case where the singular space is $\mathbb{C}^2\times\mathbb{C}^2/\Gamma$), the holomorphic constraints $\mathscr{P}_i(y)$ are identified with the holomorphic part of the tri-holomorphic moment map. When one goes to the case of multiple $M2$-branes the gauge group is enlarged by color indices. This is another story. The first step is to understand the case of one $M2$-brane and here the map between Physics and Mathematics is one-to-one.

6.1 Representations of the quivers and Kähler quotients

Let us now follow the same steps of the Kronheimer construction and derive the representations of the $\mathbb{C}^3/\Gamma$ quivers. The key point is the construction of the analogues of the spaces $\mathscr{P}_\Gamma$ in eq. (5.9) and of its invariant subspace $\mathscr{S}_\Gamma$ in eq. (5.10). To this effect we introduce three matrices $|\Gamma|\times|\Gamma|$ named A, B, C and set:

$$p \in \mathscr{P}_\Gamma \equiv \mathrm{Hom}\,(R, \mathscr{Q}\otimes R) \Rightarrow p = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{6.5}$$

The action of the discrete group Γ on the space $\mathscr{P}_\Gamma$ is defined in full analogy with the Kronheimer case:

$$\forall\gamma\in\Gamma: \quad \gamma\cdot p \equiv \mathscr{Q}(\gamma) \begin{pmatrix} R(\gamma)A\,R(\gamma^{-1}) \\ R(\gamma)B\,R(\gamma^{-1}) \\ R(\gamma)C\,R(\gamma^{-1}) \end{pmatrix} \tag{6.6}$$

where $\mathscr{Q}(\gamma)$ denotes the three-dimensional complex representation of the group element γ, while $R(\gamma)$ denotes its $|\Gamma|\times|\Gamma|$-matrix image in the regular representation.

In complete analogy with eq. (5.10) the subspace $\mathscr{S}_\Gamma$ is obtained by setting:

$$\mathscr{S}_\Gamma \equiv \mathrm{Hom}_\Gamma\,(R, Q\otimes R) = \{p\in\mathscr{P}_\Gamma / \forall\gamma\in\Gamma, \gamma\cdot p = p\} \tag{6.7}$$

Just as in the previous case a three-vector of matrices can be thought as a matrix of three-vectors: that is, $\mathscr{P}_\gamma = \mathscr{Q}\otimes\mathrm{Hom}(R,R) = \mathrm{Hom}(R, \mathscr{Q}\otimes R)$. Decomposing the regular representation, $R = \bigoplus_{i=0}^r n_i D_i$ into irreps, using eq. (6.1) and Schur's lemma, we obtain:

$$\mathscr{S}_\Gamma = \bigoplus_{i,j} A_{i,j}\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \tag{6.8}$$

The properties (6.2,6.3,6.4) of the matrix A_{ij} associated with the quiver diagram guarantee, in perfect analogy with eq. (5.13)

$$\dim_{\mathbb{C}} \mathscr{S}_\Gamma \simeq \mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R) = 3 \sum_i n_i^2 = 3|\Gamma| \,. \tag{6.9}$$

6.2 The quiver Lie group, its maximal compact subgroup and the Kähler quotient

We address now the most important point, namely the reduction of the $3|\Gamma|$-dimensional complex manifold $\mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R)$ to a $|\Gamma|+2$-dimensional subvariety of which we will perform the Kähler quotient in order to obtain the final 3-dimensional (de-singularized) smooth manifold that provides the crepant resolution. The inspiration about how this can be done is provided by comparison with the $\mathbb{C}^2/\Gamma$ case, *mutatis mutandis*. The key formulae to recall are the following ones: eq. (5.23), (5.27) and (5.34).

From eq. (5.23) we see that the analytic part of the triholomorphic moment map is provided by the projection onto the gauge group generators of the commutator $[A\,,B]$. When the level parameters are all zero (namely when the locus equation is not perturbed by the elements of the chiral ring) the outcome of the moment map equation is simply the condition $[A\,,B]=0$. In the case of $\mathbb{C}^3/\Gamma$ we already know that there are no deformations of the complex structure and that the analogue of the holomorphic moment map constraint has to be a rigid parameterless condition. Namely the ideal that cuts out the $\mathbb{V}_{|\Gamma|+2}$ variety should be generated by a list of quadric polynomials $\mathscr{P}_i(y)$ fixed once and for all in a parameterless way. It is reasonable to guess that these equations should be a generalization of the condition $[A\,,B]=0$. In the $\mathbb{C}^3/\Gamma$ case we have three matrices A,B,C and the obvious generalization is given below:

$$\mathbf{p} \wedge \mathbf{p} = 0 \tag{6.10}$$

where:

$$\begin{aligned} \mathbf{p} &= \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R) \\ p_1 &= A \quad ; \quad p_2 = B \quad ; \quad p_3 = C \end{aligned} \tag{6.11}$$

This is a short-hand for the following explicit equations

$$\begin{aligned} 0 &= \varepsilon^{ijk} \mathbf{p}_i \cdot \mathbf{p}_j \\ &\Updownarrow \\ 0 &= [A,B] = [B,C] = [C,A] \end{aligned} \tag{6.12}$$

Eq. (6.10) is the very same equation numbered (1.18) in Craw's doctoral thesis [37]. We will see in a moment that it is indeed the correct equation reducing $\mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R)$ to a $|\Gamma|+2$-dimensional subvariety. The way to understand it goes once again through a detailed comparison with the Kronheimer case.

One has to discuss the construction of the gauge group and to recall the identification of the singular orbifold $\mathbb{C}^2/\Gamma$ with the subspace named L defined by eq. (5.34). Both constructions have a completely parallel analogue in the $\mathbb{C}^3/\Gamma$ case and these provide the key to understand why (6.10) is the right choice.

Before we do that let us provide the main link between the here considered mathematical constructions and the Physics of three-dimensional Chern-Simons gauge theories. To this purpose let us go back to the results of [1]. For those special $\mathscr{N}=2$ Chern-Simons gauge theories that are actually $\mathscr{N}=3$, the superpotential $\mathscr{W}$ has the form displayed below:

$$\mathscr{W} = -\frac{1}{8\alpha} \mathscr{P}_+^\Lambda \mathscr{P}_+^\Sigma \kappa_{\Lambda\Sigma} \tag{6.13}$$

where $\mathscr{P}_{+}^{\Lambda}$ denote the holomorphic parts of the triholomorphic moment maps and $\kappa_{\Lambda\Sigma}$ is the Killing metric of the gauge Lie algebra. When looking for extrema at $V = 0$ of the scalar potential, namely for classical vacua of the gauge theory, taking into account the positive definiteness of the scalar metric $g^{\alpha\beta^\star}$ of the Killing metric $\kappa_{\Lambda\Sigma}$ and of the matrix $\mathbf{m}^{\Lambda\Sigma}$ one obtains the following conditions:

$$\mathscr{P}_{3}^{\Lambda} = \zeta_{3}^{\Lambda} \tag{6.14}$$

$$\mathscr{P}_{+}^{\Lambda} = \zeta_{+}^{\Lambda} \tag{6.15}$$

where $\mathscr{P}_{3}^{\Lambda}$ denotes the real part of the tri-holomorphic moment map. In mathematical language, the above equations just define the level set $\mu^{-1}(\zeta)$ utilized in the hyperKähler quotient.

The same field theoretic mechanism is realized in a gauge theory whose scalar fields span the space $\mathscr{S}_\Gamma$ for a $\mathbb{C}^3/\Gamma$ singularity, if we introduce the following superpotential:

$$\mathscr{W} = \mathrm{Tr}\left[p_x\, p_y\, p_z\right] \varepsilon^{xyz} \tag{6.16}$$

With this choice the conditions for the vanishing of the scalar potential are indeed the Kähler moment map equations that we are going to discuss and eq. (6.10).

6.2.1 Quiver Lie groups

We are interested in determining the subgroup

$$\mathscr{G}_\Gamma \subset \mathrm{SL}(|\Gamma|, \mathbb{C}) \tag{6.17}$$

made by those elements that commute with the group Γ.

$$\mathscr{G}_\Gamma = \{g \in \mathrm{SL}(|\Gamma|, \mathbb{C}) \quad | \quad \forall \gamma \in \Gamma \, : \, [\mathrm{D_R}(\gamma)\,,\, \mathrm{D_{def}}(g)] = 0\} \tag{6.18}$$

In the above equation $D_R()$ denotes the regular representation while $\mathrm{D_{def}}$ denotes the defining representation of the complex linear group. The two representations, by construction, have the same dimension and this is the reason why equation (6.18) makes sense.

It is sufficient to impose the defining constraint for the generators of the group on a generic matrix depending on $|\Gamma|^2$ parameters: this reduces it to a specific matrix depending on $|\Gamma|$-parameters. The further condition that the matrix should have determinant one, reduces the number of free parameters to $|\Gamma| - 1$. In more abstract terms we can say that the group $\mathscr{G}_\Gamma$ has the following general structure:

$$\mathscr{G}_\Gamma = \bigotimes_{\mu=1}^{r+1} \mathrm{GL}(\mathrm{n}_\mu, \mathbb{C}) \bigcap \mathrm{SL}(|\Gamma|, \mathbb{C}) \tag{6.19}$$

This is a perfectly analogous result to that displayed in eq. (5.27) for the Kronheimer case. The difference is that there we had unitary groups while here we are talking about general linear complex groups with a holomorphic action on the quiver coordinates. The reason is that we have not yet introduced a Kähler structure on the quiver space $\mathrm{Hom}_\Gamma(R\,,\,\mathscr{Q} \otimes R)$: we do it presently and we shall realize that isometries of the constructed Kähler metric will be only those elements of $\mathscr{G}_\Gamma$ that are contained in the unitary subgroup mentioned below:

$$\mathscr{F}_\Gamma \equiv \bigotimes_{\mu=1}^{r+1} \mathrm{U}(\mathrm{n}_\mu) \bigcap \mathrm{SU}(|\Gamma|) \subset \mathscr{G}_\Gamma \tag{6.20}$$

6.2.2 The holomorphic quiver group and the reduction to $V_{|\Gamma|+2}$

Yet the group $\mathscr{G}_\Gamma$ plays an important role in understanding the rationale of the holomorphic constraint (6.10). The key item is the coset $\mathscr{G}_\Gamma/\mathscr{F}_\Gamma$.

Let us introduce some notations. Relaying on eq. (6.5) we define the diagonal embedding:

$$\mathbb{D} \quad : \quad \mathrm{GL}(|\Gamma|,\mathbb{C}) \to \mathrm{GL}(3|\Gamma|,\mathbb{C}) \tag{6.21}$$

$$\forall M \in \mathrm{GL}(|\Gamma|,\mathbb{C}) \quad ; \quad \mathbb{D}[M] \equiv \left(\begin{array}{c|c|c} M & 0 & 0 \\ \hline 0 & M & 0 \\ \hline 0 & 0 & M \end{array}\right) \tag{6.22}$$

In this notation, the invariance condition that defines $\mathscr{S}_\Gamma = \mathrm{Hom}_\Gamma(R, \mathscr{Q} \times R)$ can be rephrased as follows:

$$\forall \gamma \in \Gamma \quad : \quad \mathscr{Q}[\gamma]\,\mathbf{p} = \mathbb{D}[R_\gamma^{-1}]\,\mathbf{p}\,\mathbb{D}[R_\gamma] \tag{6.23}$$

It is clear that any $|\Gamma| \times |\Gamma|$ - matrix M that commutes with R_γ realizes an automorphism of the space $\mathscr{S}_\Gamma$, namely it maps it into itself. The group $\mathscr{G}_\Gamma$ is such an automorphism group. In particular equation (6.10) or alternatively (6.12) is invariant under the action of $\mathscr{G}_\Gamma$. Hence the locus:

$$\begin{aligned} \mathscr{D}_\Gamma \quad &\subset \quad \mathscr{S}_\Gamma \\ \mathscr{D}_\Gamma \quad &\equiv \quad \{\mathbf{p} \in \mathscr{S}_\Gamma \mid [A,B] = [B,C] = [C,A] = 0\} \end{aligned} \tag{6.24}$$

is invariant under the action of $\mathscr{G}_\Gamma$. A priori the locus $\mathscr{D}_\Gamma$ might be empty, but this is not so because there exists an important solution of the constraint (6.10) which is the obvious analogue of the space L_Γ defined for the $\mathbb{C}^2/\Gamma$-case in eq. (5.34). In full analogy we set:

$$\mathscr{S}_\Gamma \supset L_\Gamma \equiv \left\{ \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \in \mathscr{S}_\Gamma \mid A_0, B_0, C_0 \ \text{are diagonal in the natural basis of R} : \{e_\delta\} \right\} \tag{6.25}$$

Obviously diagonal matrices commute among themselves and they do the same in any other basis where they are not diagonal, in particular in the *split basis*. By definition we name in this way the basis where the regular representation R is split into irreducible representations. A general result in finite group theory tells us that every n_i-dimensional irrep $\boldsymbol{D}_i$ appears in R exactly n_i-times:

$$R = \bigoplus_{i=0}^{r} n_i \boldsymbol{D}_i \quad ; \quad \dim \boldsymbol{D}_i \equiv n_i \tag{6.26}$$

In the split basis every element $\gamma \in \Gamma$ is given by a block diagonal matrix of the following form:

$$
R(\gamma) = \left(\begin{array}{c|c|c|c|c|c}
\mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \\
\hline
\mathbf{0} & \begin{matrix} a_{1,1} & \cdots & a_{1,n_1} \\ \vdots & \cdots & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,n_1} \end{matrix} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\
\hline
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\hline
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\hline
\mathbf{0} & \cdots & \cdots & \mathbf{0} & \begin{matrix} b_{1,1} & \cdots & b_{1,n_{r-1}} \\ \vdots & \cdots & \vdots \\ b_{n_{r-1},1} & \cdots & b_{n_{r-1},n_{r-1}} \end{matrix} & \mathbf{0} \\
\hline
\mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \begin{matrix} c_{1,1} & \cdots & c_{1,n_r} \\ \vdots & \cdots & \vdots \\ c_{n_r,1} & \cdots & c_{n_r,n_r} \end{matrix}
\end{array}\right) \tag{6.27}
$$

In appendix D we provide the explicit form of the matrices A_0,B_0,C_0 in the split basis and for the case of several groups Γ. In analogy to what was noticed for the Kronheimer case, the space L_Γ has complex dimension three (in Kronheimer case it was two):

$$
\dim_{\mathbb{C}} L_\Gamma = 3 \tag{6.28}
$$

Indeed if we fix the first diagonal entry of each of the three matrices, the invariance condition (6.23) determines all the other ones uniquely. In any other basis the number of parameters remains three. Let us call them (a_0, b_0, c_0). Because of the above argument and, once again, in full analogy with the Kronheimer case, we can conclude that the space L_Γ is isomorphic to the singular orbifold $\mathbb{C}^3/\Gamma$, the Γ-orbit of a triple (a_0, b_0, c_0) representing a point in $\mathbb{C}^3/\Gamma$.

The existence of the solution of the constraint (6.10) provided by the complex three-dimensional space L_Γ shows that we can construct a variety of dimension $|\Gamma|+2$ which is in the kernel of the constraint (6.10). This is just the orbit, under the action of $\mathscr{G}_\Gamma$ of L_Γ. We set:

$$
\mathscr{D}_\Gamma \equiv \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma) \tag{6.29}
$$

The counting is easily done.

1. A generic point in L_Γ has the identity as stability subgroup in $\mathscr{G}_\Gamma$.

2. The group $\mathscr{G}_\Gamma$ has complex dimension $|\Gamma|-1$, hence we get:

$$
\dim_{\mathbb{C}}(\mathscr{D}_\Gamma) = |\Gamma| - 1 + 3 = |\Gamma| + 2 \tag{6.30}
$$

In the sequel we define the variety $V_{|\Gamma|+2}$ to be equal to $\mathscr{D}^0_\Gamma$.

6.2.3 The coset $\mathscr{G}_\Gamma/\mathscr{F}_\Gamma$ and the Kähler quotient

It is now high time to introduce the Kähler potential of the original $3|\Gamma|$-dimensional complex flat manifold $\mathscr{S}_\Gamma$. We set:

$$\mathscr{K}_{\mathscr{S}_\Gamma} \equiv \mathrm{Tr}\left(\mathbf{p}^\dagger\,\mathbf{p}\right) = \mathrm{Tr}\left(A^\dagger A\right) + \mathrm{Tr}\left(B^\dagger B\right) + \mathrm{Tr}\left(C^\dagger C\right) \tag{6.31}$$

Using the matrix elements of A,B,C as complex coordinates of the manifold and naming λ_i the independent parameters from which they depend in a given explicit solution of the invariance constraint, the Kähler metric is defined, as usual, by:

$$ds^2_{\mathscr{S}_\Gamma} = g_{\ell\overline{m}}\,d\lambda^\ell \otimes d\overline{\lambda}^{\overline{m}} \tag{6.32}$$

where:

$$g_{\ell\overline{m}} = \partial_\ell\,\overline{\partial}_{\overline{m}}\,\mathscr{K} \tag{6.33}$$

From eq. (6.31) we easily see that the Kähler potential is invariant under the unitary subgroup of the quiver group defined by:

$$\mathscr{F}_\Gamma = \left\{M \in \mathscr{G}_\Gamma \mid M M^\dagger = \mathbf{1}\right\} \tag{6.34}$$

whose structure was already mentioned in eq. (6.20). The center $\mathfrak{z}\left(\mathbb{F}_\Gamma\right)$ of the Lie algebra $\mathbb{F}_\Gamma$ has dimension r, namely the same as the number of nontrivial conjugacy classes of Γ and it has the following structure:

$$\mathfrak{z}\left(\mathbb{F}_\Gamma\right) = \underbrace{\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)}_{r} \tag{6.35}$$

In the appendices we provide the explicit form of $\mathbb{F}_\Gamma$ while working out examples.

Since $\mathscr{F}_\Gamma$ acts as a group of isometries on the space $\mathscr{S}_\Gamma$ we might construct the Kähler quotientof the latter with respect to the former, yet we may do better.

In the case of an abelian $|\Gamma|$ the center $\mathfrak{z}[\mathbb{F}] = \mathbb{F}$ coincides with the entire gauge algebra. We discuss in detail these cases in the sequel.

Let us consider the inclusion map of the variety $\mathscr{D}_\Gamma$ into $\mathscr{S}_\Gamma$:

$$\iota\,:\,\mathscr{D}_\Gamma \to \mathscr{S}_\Gamma \tag{6.36}$$

and let us define as Kähler potential and Kähler metric of the locus $\mathscr{D}_\Gamma$ the pull backs of the Kähler potential (6.31) and of metric (6.32) of $\mathscr{S}_\Gamma$, namely let us set:

$$\mathscr{K}_{\mathscr{D}_\Gamma} \equiv \iota^\star\,\mathscr{K}_{\mathscr{S}_\Gamma} \tag{6.37}$$

$$ds^2_{\mathscr{D}_\Gamma} = \iota^\star\,ds^2_{\mathscr{S}_\Gamma} \tag{6.38}$$

By construction, the isometry group $\mathscr{F}_\Gamma$ is inherited by the pullback metric on $\mathscr{D}_\Gamma$ and we can consider the Kähler quotient:

$$\mathscr{M}_\zeta \equiv \mathscr{D}_\Gamma /\!/^{\zeta}_{\mathscr{F}_\Gamma} \tag{6.39}$$

Let f_I be a basis of generators of $\mathscr{F}_\Gamma$ ($I = 1,\ldots,|\Gamma|-1$) and let Z_i ($i = 1,\ldots,|\Gamma|+2$) be a system of complex coordinates spanning the points of $\mathscr{D}_\Gamma$. By means of the inclusion map we have:

$$\forall Z \in \mathscr{D}_\Gamma \quad : \quad \iota(Z) = \mathbf{p}(Z) = \begin{pmatrix} A(Z) \\ B(Z) \\ C(Z) \end{pmatrix} \tag{6.40}$$

The action of the *gauge group* $\mathscr{F}_\Gamma$ on $\mathscr{D}_\Gamma$ is implicitly defined by:

$$\mathbf{p}(\delta_I Z) = \delta_I \mathbf{p}(Z) = \begin{pmatrix} [f_I, A(Z)] \\ [f_I, B(Z)] \\ [f_I, C(Z)] \end{pmatrix} \tag{6.41}$$

and the corresponding real moment maps are easily calculated:

$$\mu_I(Z,\overline{Z}) = \mathrm{Tr}\left(f_I\left[A(Z),A^\dagger(\overline{Z})\right]\right) + \mathrm{Tr}\left(f_I\left[B(Z),B^\dagger(\overline{Z})\right]\right) + \mathrm{Tr}\left(f_I\left[C(Z),C^\dagger(\overline{Z})\right]\right) \tag{6.42}$$

One defines the level sets by means of the equation:

$$\mu^{-1}(\zeta) = \left\{Z \in \mathscr{D}_\Gamma \parallel \mu_I(Z,\overline{Z}) = 0 \quad \text{if } f_I \notin \mathfrak{Z} \quad ; \quad \mu_I(Z,\overline{Z}) = \zeta_I \quad \text{if } f_I \in \mathfrak{Z}\right\} \tag{6.43}$$

which, by construction, are invariant under the gauge group $\mathscr{F}_\Gamma$ and we can finally set:

$$\mathscr{M}_\zeta \equiv \mu^{-1}(\zeta) /\!/_{\mathscr{F}_\Gamma} \equiv \mathscr{D}_\Gamma /\!/^{\zeta}_{\mathscr{F}_\Gamma} \tag{6.44}$$

The real and complex dimensions of $\mathscr{M}_\zeta$ are easily calculated. We start from $|\Gamma|+2$ complex dimensions, namely from $2|\Gamma|+4$ real dimensions. The level set equation imposes $|\Gamma|-1$ real constraints, while the quotiening by the group action takes other $|\Gamma|-1$ parameters away. Altogether we remain with 6 real parameters that can be seen as 3 complex ones. Hence the manifolds $\mathscr{M}_\zeta$ are always complex three-folds that, for generic values of ζ, are smooth: supposedly the crepant resolutions of the singular orbifold. For $\zeta = 0$ the manifold $\mathscr{M}_0$ degenerates into the singular orbifold $\mathbb{C}^3/\Gamma$, since the solution of the moment map equation is given by the $\mathscr{F}_\Gamma$ orbit of the locus L_Γ, namely:

$$\mu^{-1}(0) = \mathrm{Orbit}_{\mathscr{F}_\Gamma}(L_\Gamma) \tag{6.45}$$

Comparing eq. (6.29) with eq. (6.45) we are led to consider the following direct sum decomposition of the Lie algebra:

$$\mathbb{G}_\Gamma = \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma \tag{6.46}$$

$$[\mathbb{F}_\Gamma, \mathbb{F}_\Gamma] \subset \mathbb{F}_\Gamma \quad ; \quad [\mathbb{F}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{K}_\Gamma \quad ; \quad [\mathbb{K}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{F}_\Gamma \tag{6.47}$$

where $\mathbb{F}_\Gamma$ is the maximal compact subalgebra and $\mathbb{K}_\Gamma$ denotes its complementary orthogonal subspace with respect to the Cartan Killing metric.

A special feature of all the quiver Groups and Lie Algebras is that $\mathbb{F}_\Gamma$ and $\mathbb{K}_\Gamma$ have the same real dimension $|\Gamma|-1$ and one can choose a basis of hermitian generators T_I such that:

$$\begin{array}{lcccl} \forall \mathbf{\Phi} \in \mathbb{F}_\Gamma & : & \mathbf{\Phi} & = & \mathrm{i} \times \sum_{I=1}^{|\Gamma|-1} c_I T^I \quad ; \quad c_I \in \mathbf{R} \\ \forall \boldsymbol{K} \in \mathbb{K}_\Gamma & : & \boldsymbol{K} & = & \sum_{I=1}^{|\Gamma|-1} b_I T^I \quad ; \quad b_I \in \mathbf{R} \end{array} \tag{6.48}$$

Correspondingly a generic element $g \in \mathscr{G}_\Gamma$ can be split as follows:

$$\forall g \in \mathscr{G}_\Gamma \quad : \quad g = \mathscr{U}\,\mathscr{H} \quad ; \quad \mathscr{U} \in \mathscr{F}_\Gamma \quad ; \quad \mathscr{H} \in \exp[\mathbb{K}_\Gamma] \tag{6.49}$$

Using the above property we arrive at the following parametrization of the space $\mathscr{D}_\Gamma$

$$\mathscr{D}_\Gamma = \mathrm{Orbit}_{\mathscr{F}_\Gamma}(\exp[\mathbb{K}_\Gamma] \cdot L_\Gamma) \tag{6.50}$$

where, by definition, we have set:

$$
\begin{aligned}
p \in \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \quad &\Rightarrow \quad p = \{\exp[-\boldsymbol{K}]\, A_0 \exp[\boldsymbol{K}]\,, \exp[-\boldsymbol{K}]\, B_0 \exp[\boldsymbol{K}]\,, \exp[-\boldsymbol{K}]\, C_0 \exp[\boldsymbol{K}]\} && (6.51)\\
\{A_0, B_0, C_0\} \quad &\in \quad L_\Gamma && (6.52)\\
\boldsymbol{K} \quad &= \quad \mathbb{K}_\Gamma && (6.53)
\end{aligned}
$$

Relying on this, in the Kähler quotient we can invert the order of the operations. First we quotient the action of the compact gauge group $\mathscr{F}_\Gamma$ and then we implement the moment map constraints. We have:

$$
\mathscr{D}_\Gamma /\!/ _{\mathscr{F}_\Gamma} = \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \tag{6.54}
$$

Calculating the moment maps on $\exp[\mathbb{K}_\Gamma] \cdot L_\Gamma$ and imposing the moment map constraint we find:

$$
\mu^{-1}(\zeta) /\!/ _{\mathscr{F}_\Gamma} = \left\{ Z \in \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \parallel \mu_I(Z, \overline{Z}) = 0 \quad \text{if } f_I \notin \mathfrak{Z} \quad ; \quad \mu_I(Z, \overline{Z}) = \zeta_I \quad \text{if } f_I \in \mathfrak{Z} \right\} \tag{6.55}
$$

Eq. (6.55) provides an explicit algorithm to calculate the Kähler potential of the final resolved manifold if we are able to solve the constraints in terms of an appropriate triple of complex coordinates. Furthermore for each level parameter ζ_a we have to find the appropriate one-parameter subgroup of $\mathscr{G}_\Gamma$ that lifts the corresponding moment map from the 0-value to the generic value ζ. Indeed we recall that the Kähler potential of the resolved variety is given by the celebrated formula:

$$
\mathscr{K}_{\mathscr{M}} = \pi^\star \mathscr{K}_{\mathscr{N}} + \zeta_I \mathfrak{C}^{IJ} \boldsymbol{\Phi}_J \tag{6.56}
$$

where, by definition:

$$
\pi : \mathscr{N} \to \mathscr{M} \tag{6.57}
$$

is the quotient map and $\exp[\zeta_I \mathfrak{C}^{IJ} \boldsymbol{\Phi}_I] \in \exp[\mathbb{K}_\Gamma] \subset \mathscr{G}_\Gamma$ is the element of the quiver group which lifts the moment maps from zero to the values ζ_I, while $\mathfrak{C}^{IJ}$ is a constant matrix whose definition we discuss later on. Indeed the rationale behind formula (6.56) requires a careful discussion, originally due to Hitchin, Karlhede, Lindström and Roček [42] which we shall review in the next section.

7 Lessons from the Eguchi-Hanson case

In order to give a concrete illustrative example of the Kronheimer construction we focus on the simplest and oldest known ALE manifold, namely on the Eguchi-Hanson space [43]. To this effect we begin by introducing a set of Maurer Cartan forms on the three sphere $\mathbb{S}^3 \sim \mathrm{SU}(2)$:

$$
\begin{aligned}
\sigma_1 &= -\frac{1}{2}(d\theta \cos(\psi) + d\phi \sin(\theta) \sin(\psi))\\
\sigma_2 &= \frac{1}{2}(d\theta \sin(\psi) - d\phi \sin(\theta) \cos(\psi))\\
\sigma_3 &= -\frac{1}{2}(d\phi \cos(\theta) + d\psi)
\end{aligned} \tag{7.1}
$$

which depend on three Euler angles θ, ϕ, ψ and satisfy the Maurer Cartan equations in the form:

$$
d\sigma_i = \varepsilon_{ijk}\, \sigma_j \wedge \sigma_k \tag{7.2}
$$

Furthermore, let us introduce a radial coordinate $m \leq r \leq +\infty$ and the following function:

$$
G(r) = \sqrt{1 - \left(\frac{m}{r}\right)^2} \tag{7.3}
$$

The Eguchi-Hanson metric is given by the following expression:

$$
\begin{aligned}
ds^2_{EH} &= G(r)^{-2}\,dr^2 + r^2\left(\sigma_1^2+\sigma_2^2\right) + r^2\,G(r)^2\,\sigma_3^2 \\
&= \frac{1}{4}\left(\frac{\left(r^4-a^4\right)\left(d\phi\cos(\theta)+d\psi\right)^2}{r^2}+\frac{4dr^2}{1-\frac{a^4}{r^4}}+r^2\left(d\phi^2\sin^2(\theta)+d\theta^2\right)\right)
\end{aligned}
\tag{7.4}
$$

Calculating the curvature two-form of the above metric, we find that it is self-dual, while its Ricci tensor vanishes. Hence the Eguchi-Hanson metric is an Euclidean vacuum solution of Einstein equations and it describes a gravitational instanton. As $r \to \infty$ the Eguchi-Hanson metric approaches a Euclidean metric:

$$
ds^2_{EH} \overset{r\to\infty}{\Longrightarrow} \frac{1}{2}r^2 d\psi d\phi\cos(\theta)+\frac{1}{4}r^2d\theta^2+\frac{1}{4}r^2d\psi^2+\frac{1}{4}r^2d\phi^2+dr^2 \tag{7.5}
$$

Next, one describes the Eguchi-Hanson space as a complex manifold $\mathscr{M}_{EH}$ and the Eguchi-Hanson metric $\widehat{ds^2_{EH}}$ as a Kähler metric on $\mathscr{M}_{EH}$. To this effect let us introduce the following two complex coordinates:

$$
\begin{aligned}
Z^1 &= \left(r^4-m^4\right)^{\frac{1}{4}}\,\frac{\left(e^{i(\theta+\phi)}+ie^{i\theta}+e^{i\phi}-i\right)e^{-\frac{1}{2}i(\theta-\psi+\phi)}}{2\sqrt{2}} \\
Z^2 &= \left(r^4-m^4\right)^{\frac{1}{4}}\,\frac{\left(e^{i(\theta+\phi)}-ie^{i\theta}+e^{i\phi}+i\right)e^{-\frac{1}{2}i(\theta-\psi+\phi)}}{2\sqrt{2}}
\end{aligned}
\tag{7.6}
$$

By direct calculation we can verify that:

$$
ds^2_{EH} = \frac{\partial}{\partial Z^i}\frac{\partial}{\partial \overline{Z}^{j^\star}}\,\mathscr{K}_{EH}\,dZ^i\otimes d\overline{Z}^{j^\star} \tag{7.7}
$$

where:

$$
\begin{aligned}
\mathscr{K}_{EH} &= \sqrt{\tau^2+m^4}-m^2\,\log\left(\sqrt{\tau^2+1}+m^4\right)+m^2\,\log(\tau) \\
\tau &\equiv |Z^1|^2+|Z^2|^2
\end{aligned}
\tag{7.8}
$$

Having derived the form of the Kähler potential for the Eguchi-Hanson metric we can now connect it to the Kronheimer construction of the ALE manifolds by recalling eqs. (5.32) and rewriting them in the case $k=1$ for which the group $\mathscr{F} = \mathrm{U}(1)$, so that there is only one component of the triholomorphic moment map:

$$
\mathfrak{P}^3 = |u_0|^2-|v_0|^2+|v_1|^2-|u_1|^2 \tag{7.9}
$$

$$
\mathfrak{P}^+ = u^0\,v_0-u^1\,v_1 \tag{7.10}
$$

In this case it is convenient to redefine:

$$
U = \{u_0, v_1\} \tag{7.11}
$$

$$
V = \{v_0, u_1\} \tag{7.12}
$$

so that eqs. (7.10) can be rewritten as follows:

$$\mathfrak{P}^3 = \mathscr{P}^3(U,V) \equiv \sum_{i=1}^{2} |U_i|^2 - \sum_{i=1}^{2} |V_i|^2 \tag{7.13}$$

$$\mathfrak{P}^+ = \mathscr{P}^+(U,V) \equiv \sum_{i=1}^{2} U_i V_i \tag{7.14}$$

Furthermore the action of the group $\mathscr{F}_{\mathbb{Z}_2} = \mathrm{U}(1)$ on the complex coordinates U,V is the following one:

$$\mathrm{U}(1) \quad : \quad (U,V) \quad \Longrightarrow \quad \left(e^{i\varphi}U\,,\,e^{-i\varphi}V\right) \tag{7.15}$$

Considering the quiver group $\mathscr{G}_{\mathbb{Z}_2}$ which is just the complexification of $\mathscr{F}_{\mathbb{Z}_2}$ we obtain the transformation:

$$\mathscr{G}_{\mathbb{Z}_2} \quad : \quad (U,V) \quad \Longrightarrow \quad \left(e^{-\mathbf{\Phi}}U\,,\,e^{\mathbf{\Phi}}V\right) \tag{7.16}$$

Relying on these preliminaries we are ready to perform the algebro-geometric hyperKähler quotient according to formula (6.56). Introducing the level parameters we have to solve the equations:

$$\begin{aligned} \ell &= \mathscr{P}^3\left(e^{-\mathbf{\Phi}}U, e^{\mathbf{\Phi}}V\right) \\ \mathfrak{s} &= \mathscr{P}^+\left(e^{-\mathbf{\Phi}}U, e^{\mathbf{\Phi}}V\right) = \mathscr{P}^+(U,V) \end{aligned} \tag{7.17}$$

As stated several times and recalled in the second line of the above equation, the holomorphic part of the moment map is invariant under the action of the quiver group. This is very useful for the solution of the constraints. Indeed we can just choose a gauge condition like the following one:

$$U_1 = V_2 \tag{7.18}$$

Furthermore, in the case $k=1$ the holomorphic level parameter $\mathfrak{s}$ can be just set equal to zero without loss of generality, since it simply amounts to a change of coordinates. In this way we arrive at:

$$U_1 = V_2 \equiv \tfrac{1}{2}Z^1 \quad ; \quad U_2 = V_1 \equiv \tfrac{1}{2}Z^2 \tag{7.19}$$

and the first of equations (7.17) is solved by:

$$\mathbf{\Phi} = -\log\left[\frac{\ell \pm \sqrt{\ell^2 + 4|U|^2|V^2|}}{2|V^2|}\right] = -\log\left[\frac{\ell \pm \sqrt{\ell^2 + |\mathbf{Z}|^4}}{2|\mathbf{Z}|^2}\right] \quad ; \quad |\mathbf{Z}|^2 \equiv |Z^1|^2 + |Z^2|^2 \tag{7.20}$$

The restriction to the level surface of the ambient Kähler potential is calculated in an equally easy fashion:

$$\mathscr{K}|_{\mathscr{N}} = e^{-2\mathbf{\Phi}}|U|^2 + e^{2\mathbf{\Phi}}|V|^2 = \sqrt{\ell^2 + |\mathbf{Z}|^4} \tag{7.21}$$

Choosing one branch of the solution (7.20) and applying the general formula (6.56) we obtain the Kähler potential of the manifold $\mathscr{M}$:

$$\mathscr{K}_{\mathscr{M}} = \sqrt{\ell^2 + |\mathbf{Z}|^4} - \ell \log\left[\frac{\ell \pm \sqrt{\ell^2 + |\mathbf{Z}|^4}}{2|\mathbf{Z}|^2}\right] \tag{7.22}$$

For $\ell = m^2$, we see that the Kähler potential (7.22) obtained by means of the hyperKähler quotient advocated in the Kronheimer construction coincides with that of the Eguchi-Hanson manifold displayed in eq. (7.8). This concludes the proof that the Eguchi-Hanson manifold is a smooth resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_2$.

7.1 The algebraic equation of the locus and the exceptional divisor

First we consider the algebraic equation of the locus in $\mathbb{C}^3$ that corresponds to the Eguchi-Hanson manifold. According to the discussion following eq. (5.39) such an equation is provided by the relation between the Γ invariants:

$$x \equiv \mathrm{Det} A \quad ; \quad \mathrm{Det} B \quad ; \quad z \equiv \tfrac{1}{2}\,\mathrm{Tr}\,(A\,B) \tag{7.23}$$

Upon use of the gauge condition (7.18) and of the solution of the holomorphic moment map constraint (7.19) we have:

$$A = \left(\begin{array}{cc} 0 & \frac{1}{2}Z_1 \\ \frac{1}{2}Z_2 & 0 \end{array}\right) \quad ; \quad B = \left(\begin{array}{cc} 0 & \frac{1}{2}Z_2 \\ \frac{1}{2}Z_1 & 0 \end{array}\right) \tag{7.24}$$

so that:

$$x = -\tfrac{1}{4}Z_1\,Z_2 \quad ; \quad y = -\tfrac{1}{4}Z_1\,Z_2 \quad ; \quad z = \tfrac{1}{4}Z_1\,Z_2 \tag{7.25}$$

and the equation of the orbifold locus $\mathbb{C}^2/\Gamma$:

$$xy = z^2 \tag{7.26}$$

remains unmodified. This happens because the holomorphic moment map has not been lifted away from zero and similarly will happen in all the resolutions of the $\mathbb{C}^3/\Gamma$ singularities since, as we stressed, there we have no complex structure deformations and the analogue of the holomorphic moment map equation $[A\,,B] = [B\,,C] = [C\,,A]$ obtains no deformation. Yet we know that by lifting the level of the real moment map we obtain the smooth Eguchi-Hanson manifold which has a nontrivial homology 2-cycle, as foreseen by the general theorem 4.1. In quasi polar coordinates these homology cycle is the two–sphere spanned by angles θ and ϕ when we set $r = m$ and we disregard the angle ψ. Such a homology cycle disappears when $m \to 0$ hence it is the exceptional divisor generated by the minimal resolution of the singularity. Hence it is interesting to see where such an exceptional divisor is located in the complex description of the Eguchi-Hanson manifold obtained from the Kronheimer construction. To this effect it is convenient to recall the relation between divisors and line bundles.

7.1.1 Divisors and line bundles

A *prime divisor* in a complex manifold or algebraic variety X is an irreducible closed codimension one subvariety of X. A divisor $\mathfrak{D}$ is a locally finite formal linear combination

$$\mathfrak{D} = \sum_i a_i\,\mathfrak{D}_i \tag{7.27}$$

where the a_i are integers, and the $\mathfrak{D}_i$ are prime divisors. A prime divisor $\mathfrak{D}$ can be descrived by a collection $\{(U_\alpha, f_\alpha)\}$, where $\{U_\alpha\}$ is an open cover of X, and the $\{f_\alpha\}$ are holomorphic functions on U_α such that $f_\alpha = 0$ is the equation of $\mathfrak{D} \cap U_\alpha$ in U_α. As a consequence, the functions $g_{\alpha\beta} = f_\alpha/f_\beta$ are holomorphic nowhere vanishing functions

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^*$$

that on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ satisfy the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

and therefore define a line bundle $\mathscr{L}(\mathfrak{D})$. If $\mathfrak{D}$ is a divisor as in (7.27) then one sets

$$\mathscr{L}(\mathfrak{D}) = \bigotimes_i \mathscr{L}(\mathfrak{D}_i)^{a_i}.$$

The inverse correspondence (from line bundles to divisors) is described as follows. If s is a nonzero meromorphic section of a line bundle $\mathscr{L}$, and V is a codimension one subvariety of X over which s has a zero or a pole, denoted by $\mathrm{ord}_V(s)$ the order of the zero, or minus the order of the pole; then

$$\mathfrak{D} = \sum_V \mathrm{ord}_V(s) \cdot V$$

is a divisor, whose associated line bundle $\mathscr{L}(\mathfrak{D})$ is isomorphic to $\mathscr{L}$.

7.1.2 The exceptional divisor

It is easy to work out the exceptional divisor in the Eguchi-Hanson case by performing the following holomorphic coordinate transformation:

$$Z^1 \to (1-\xi_1)\,\xi_2 \quad ; \quad Z^2 \to -(1+\xi_1)\,\xi_2 \tag{7.28}$$

Upon the substitution (7.28) and the identification $\ell = m^2$ the Kähler potential (7.22) becomes:

$$\begin{aligned}
\mathscr{K}_{EH} &= \mathscr{K}_0 + m^2\left(\mathscr{K}_\mathrm{E} + \log|W|^2\right) \\
\mathscr{K}_0 &= \sqrt{m^4 + 4\,(1+|\xi_1|^2)^2\,|\xi_2|^4} - m^2 \log\left(m^2 + \sqrt{m^4 + 4\,(1+|\xi_1|^2)^2\,|\xi_2|^4}\right) \\
\mathscr{K}_\mathrm{E} &= \log\left(1+|\xi_1|^2\right) \\
W &\equiv \sqrt{2}\,\xi_2
\end{aligned} \tag{7.29}$$

Inspecting eq. (7.29) we realize that $\mathscr{K}_\mathrm{E}$ is the standard Kähler potential of a $\mathbb{P}^1$ written in the affine coordinate ξ_1. This suggests that the Eguchi-Hanson manifold is covered by two open charts:

$$\begin{aligned}
U_N &= \left\{\xi_1^N, \xi_2^N\right\} \\
U_S &= \left\{\xi_1^S, \xi_2^S\right\}
\end{aligned} \tag{7.30}$$

with transition function:

$$\left\{\xi_1^N, \xi_2^N\right\} = \left\{\frac{1}{\xi_1^S}, \xi_2^S\,\xi_1^S\right\} \tag{7.31}$$

Under the transformation (7.30) the function $\mathscr{K}_0$ is invariant, while $\mathscr{K}_\mathrm{E}$ transforms as follows:

$$\mathscr{K}_\mathrm{E}\left(\xi^N, \overline{\xi^N}\right) = \mathscr{K}_\mathrm{E}\left(\xi^S, \overline{\xi^S}\right) - \log|\xi_1^S|^2 \tag{7.32}$$

Therefore we can introduce a line bundle $\mathscr{L} \xrightarrow{\pi} \mathscr{M}_{EH}$ defined by two local trivializations, one based on U_N, the other on U_S with transition function:

$$g_{NS} \quad : \quad W_N(\xi^N) = \xi_1^S W_S(\xi^S) \tag{7.33}$$

a fiber metric on such a bundle is obtained by defining the following invariant norm for the bundle sections:

$$\parallel W \parallel^2 \equiv e^{\mathscr{K}_\mathrm{E}}\,|W|^2 \tag{7.34}$$

The corresponding first Chern class is:

$$c_1(\mathscr{L}) = \omega^{(1,1)} \equiv \frac{i}{2\pi}\,\partial\overline{\partial}\log \parallel W \parallel^2 \quad \overset{W\to 0}{\longrightarrow} \quad \frac{i}{2\pi}\,\frac{d\xi_1 \wedge d\overline{\xi}_1}{(1+|\xi_1|^2)^2} \equiv \omega_{\mathscr{D}}^{(1,1)} \tag{7.35}$$

The divisor $\mathfrak{D}$ related with this line bundle is obviously obtained as the vanishing locus of the global section $W = \xi_2 = 0$. The cohomology class of $\omega^{(1,1)}$ is that of the Poincaré dual $\omega^{(1,1)}_{\mathscr{D}}$ of the vanishing section W, namely of the divisor $\mathfrak{D}$:

$$\left[\omega^{(1,1)}\right] = \left[\omega^{(1,1)}_{\mathfrak{D}}\right] \tag{7.36}$$

What we have discussed so far is just an explicit illustration of the well known fact that the Eguchi-Hanson manifold is the total space of the fiber bundle $\mathscr{O}_{\mathbb{P}^1}(-2)$.

Since the function $\mathscr{K}_0$ is invariant, it is clear that its contribution $\partial\overline{\partial}\mathscr{K}_0$ to the Kähler 2-form is cohomologous to zero which implies:

$$[\mathbb{K}_{EH}] = m^2\left[\omega^{(1,1)}_{\mathfrak{D}}\right] \tag{7.37}$$

Finally it is instructive to compare the above complex description of the Eguchi-Hanson space with its description in terms of quasi polar coordinates. To this effect it suffices to rewrite the coordinate transformation (7.6) in terms of the x_i coordinates. We have:

$$\xi_1 = e^{i\phi}\cot\left(\frac{\theta}{2}\right), \xi_2 = \frac{1}{2}\sqrt{1-\cos(\theta)}\sqrt[4]{r^4-m^4}e^{\frac{1}{2}i(\psi-\phi)} \tag{7.38}$$

As we see the locus $\xi_2 = 0$ corresponds to $r = m$ and $\psi =$ any value.

7.2 Comparison with the two-center Gibbons-Hawking metric

Finally we derive the map between the manifold with a two center Gibbons-Hawking (GH) metric [44, 45] and the Eguchi-Hanson space. We begin with a conceptual discussion about the parameters of GH-metrics (Appendix C is devoted to a concise description of GH-metrics and also provides new formulae concerning their explicit description in terms of complex coordinates. There we also address the problem of deriving the corresponding Kähler potential).

The Gibbons-Hawking multi-center metrics have a number of parameters that can be counted in the following way. Let n be the number of centers. Each center has 3-coordinates, hence a priori we have $3n$ parameters. Yet, using the Euclidean group of translations and rotations, which is a symmetry of the $3d$ laplacian, we can always bring a center to a reference point, say the origin $\boldsymbol{x} = 0$. So we are left with $3(n-1)$ parameters. Furthermore, once a center is fixed, another center lies somewhere on a two-sphere around the first center and we can use the rotation group to bring it to a preferred direction. This kills two additional parameters. In this way we have:

$$\#\text{ of effective parameters in a GH metric} = 3n-5 \tag{7.39}$$

From the point of view of the Kronheimer construction, the n-center metric corresponds to the resolution $Y \to \frac{\mathbb{C}^2}{\mathbb{Z}_m}$ via a hyperKähler quotient. In this case the gauge group is $\mathrm{U}(1)^{n-1}$ and we have indeed $3(n-1)$ parameters. Two parameters corresponding to one complex moment map level can be disposed of by a redefinition of the complex coordinates for the resolved manifold Y. Hence also on the side of the hyperKähler quotient we have:

$$\#\text{ of effective parameters in a hyperKähler quotient resolution of } \frac{\mathbb{C}^2}{\mathbb{Z}_n} = 3n-5 \tag{7.40}$$

In the Eguchi-Hanson case $n = 2$ and there is only one effective parameter on both sides of the correspondence, namely the parameter m^2 that we have associated with real moment map level. The level of the holomorphic moment map corresponds to the two parameters that can be disposed of by a coordinate transformation and was set to zero.

From the GH-side, the removal of the spurious parameters can be conventionally performed by aligning the two centers on the z-axis at symmetrical positions with respect to the origin $z = 0$. Hence referring to eqs.

(C.1)and (C.2) we set:

$$\mathscr{V}_{EH} = \frac{1}{\sqrt{\left(\frac{m^2}{8}+z\right)^2+x^2+y^2}} + \frac{1}{\sqrt{\left(z-\frac{m^2}{8}\right)^2+x^2+y^2}} \tag{7.41}$$

and we obtain the following connection one-form :

$$\begin{aligned} \omega_{EH} &= \left(m^2\left(\frac{1}{\sqrt{(m^2-8z)^2+64\left(x^2+y^2\right)}} - \frac{1}{\sqrt{(m^2+8z)^2+64\left(x^2+y^2\right)}}\right)\right. \\ &\quad \left. - \frac{8z}{\sqrt{(m^2-8z)^2+64\left(x^2+y^2\right)}} - \frac{8z}{\sqrt{(m^2+8z)^2+64\left(x^2+y^2\right)}} + 2\right) \times \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2+y^2} \end{aligned} \tag{7.42}$$

which satisfies with $\mathscr{V}_{EH}$ the duality relation (C.4). The one-form ω_{EH} agrees with eq. (C.17) if we set:

$$\begin{aligned} \partial_z \mathscr{F}_{EH} &= \int dz \mathscr{V}_{EH} \\ &= \log\left(\sqrt{\left(z-\frac{m^2}{8}\right)^2+x^2+y^2} - \frac{m^2}{8}+z\right) + \log\left(\sqrt{\left(\frac{m^2}{8}+z\right)^2+x^2+y^2} + \frac{m^2}{8}+z\right) \end{aligned} \tag{7.43}$$

The metric:

$$ds^2_{two-center} = \frac{1}{\mathscr{V}_{EH}}\left(\mathrm{d}\tau+\omega_{EH}\right)^2 + \mathscr{V}_{EH}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right) \tag{7.44}$$

is exactly mapped into the Eguchi-Hanson metric (7.4) by the following coordinate transformation:

$$\begin{aligned} & x \to \frac{1}{8}\sin(\theta)\sqrt{r^4-m^4}\cos(\psi) \quad , \quad y \to \frac{1}{8}\sin(\theta)\sqrt{r^4-m^4}\sin(\psi) \\ & z \to \frac{1}{8}r^2\cos(\theta) \quad , \quad \tau \to 2\psi+2\phi \end{aligned} \tag{7.45}$$

It is also interesting to work out the explicit form, in the present case of the complex coordinates $\mathfrak{h}$ and $\mathfrak{z}$ introduced in eqs. (C.16) and (C.22) within the framework of the general discussion. After some algebra one finds:

$$\mathfrak{h} = \frac{64\,e^{2i(\psi+\phi)}}{(\cos(\theta)+1)^2\,(r^4-m^4)} \quad , \quad \mathfrak{z} = \frac{1}{8}\,i\,e^{-i\psi}\sin(\theta)\sqrt{r^4-m^4} \tag{7.46}$$

As one realizes, both these coordinates are singular on the exceptional divisor $r=m$ and they are not convenient to describe it. The relation with the good coordinates $\xi_{1,2}$ is actually antiholomorphic and it would be difficult to be guessed a priori:

$$\overline{\xi}_1 = -\frac{i}{\mathfrak{z}\sqrt{\mathfrak{h}}} \quad , \quad \overline{\xi}_2 = -\frac{\sqrt{2}\,\mathfrak{z}\,\sqrt[4]{\mathfrak{h}}}{m} \tag{7.47}$$

In terms of the GH-coordinates, by inspecting eq. (7.45) we readily retrieve the image of exceptional divisor inside the GH space. It is given by the locus:

$$\mathfrak{D}_E = \left\{x=y=0\,,\,-\frac{m^2}{8} \le z \le \frac{m^2}{8}\,,\,0 \le \tau \le 2\pi\right\} \tag{7.48}$$

namely the product of the segment joining the two centers on the z-axis with the circle spanned by the τ-angle. (Actually a detailed analysis of the metric shows that it degenerates at the ends of the cylinder, so that the latter may be thought of as a sphere.) This observation may be useful in order to find the exceptional divisors in the more complicated multi-center cases.

8 The generalized Kronheimer construction for $\frac{\mathbb{C}^3}{\Gamma}$ and the Tautological Bundles

In the present section we aim at extracting a general scheme from the detailed discussions presented in the previous sections. Our final goal is to establish all the algorithmic steps that give a precise meaning to each of the lines appearing in the conceptual diagram of eq. (4.2).

8.1 Construction of the space $\mathscr{N}_\zeta \equiv \mu^{-1}(\zeta)$

Summarizing the points of our construction we have the following situation. We have considered the moment map

$$\mu : \mathscr{S}_\Gamma \longrightarrow \mathbb{F}_\Gamma{}^* \tag{8.1}$$

where $\mathbb{F}_\Gamma{}^*$ is the dual of the Lie algebra of the maximal compact subgroup $\mathscr{F}_\Gamma$ of the quiver group $\mathscr{G}_\Gamma$. Next we have considered the center of the above Lie algebra $\mathfrak{z}[\mathbb{F}_\Gamma] \subset \mathbb{F}_\Gamma$ and its dual $\mathfrak{z}[\mathbb{F}_\Gamma]^*$. The moment map can be restricted to the subspace:

$$\mathscr{D}_\Gamma \subset \mathscr{S}_\Gamma \quad ; \quad \mathscr{D}_\Gamma \equiv \{p \in \mathscr{S}_\Gamma | \, p \wedge p = 0\} \tag{8.2}$$

which is just the orbit, with respect to the quiver group $\mathscr{G}_\Gamma$, of a locus $\mathscr{E}_\Gamma \subset \mathscr{S}_\Gamma$ of complex dimension three obtained in the following way.

Consider the following subspace of $\mathscr{S}_\Gamma^{[0,0]} \subset \mathscr{S}_\Gamma$

$$\mathscr{S}_\Gamma^{[0,0]} = \{p \in \mathscr{S}_\Gamma | \, p \wedge p = 0 \quad ; \quad \mu(p) = 0\} \tag{8.3}$$

whose elements are triples of $|\Gamma| \times |\Gamma|$ complex matrices (A,B,C) satisfying, by the above definition, in addition to the invariance constraint (6.6-6.7) also the following two ones:

$$[A,B] = [B,C] = [C,A] = 0 \quad ; \quad \mathrm{Tr}\left[T_I\left(\left[A,A^\dagger\right] + \left[B,B^\dagger\right] + \left[C,C^\dagger\right]\right)\right] = 0 \quad ; \quad I = 1,\ldots,|\Gamma|-1 \tag{8.4}$$

Since the action of the compact group $\mathscr{F}_\Gamma$ leaves both the first and the second constraint invariant, it follows that it maps the locus $\mathscr{S}_\Gamma^{[0,0]}$ into itself

$$\mathscr{F}_\Gamma \quad : \quad \mathscr{S}_\Gamma^{[0,0]} \to \mathscr{S}_\Gamma^{[0,0]} \tag{8.5}$$

The locus $\mathscr{E}_\Gamma$ is defined as the quotient:

$$\mathscr{E}_\Gamma \equiv \frac{\mathscr{S}_\Gamma^{[0,0]}}{\mathscr{F}_\Gamma} \tag{8.6}$$

which turns out to be of complex dimension three and to be isomorphic to the singular orbifold :

$$\frac{\mathscr{S}_\Gamma^{[0,0]}}{\mathscr{F}_\Gamma} \simeq \frac{\mathbb{C}^3}{\Gamma} \tag{8.7}$$

Choosing a representative in each equivalence class $\frac{\mathscr{S}_\Gamma^{[0,0]}}{\mathscr{F}_\Gamma}$ simply amounts to a choice of local coordinates on $\frac{\mathbb{C}^3}{\Gamma}$ which will be promoted to a system of local coordinates on the manifold $\mathscr{M}_\zeta$ of the final resolved singularity.

We have a canonical algorithm to construct a canonical coordinate system for $\mathscr{E}_\Gamma$ which originates from Kronheimer and from the 1994 paper by Anselmi, Billò, Frè, Girardello and Zaffaroni on ALE manifolds and conformal field theories [38]. The construction is the following. We begin with the locus $L_\Gamma \subset \mathscr{S}_\Gamma$ defined as the set of triples (A_d,B_d,C_d) such that the invariance constraint (6.7) is satisfied with respect to Γ and they are diagonal in the natural basis of the regular representation. We have shown on the basis of several examples that :

$$\mathscr{D}_\Gamma = \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma) \tag{8.8}$$

We obtain an explicit parameterization of the locus $\mathscr{E}_\Gamma$ by solving the algebraic equation for the hermitian matrix $\mathscr{V}_0 \in \exp[\mathbb{K}_\Gamma]$, such that

$$\forall p \in L_\Gamma \quad : \quad \mu(\mathscr{V}_0.p) = 0 \tag{8.9}$$

The important thing is that the solution for the above equation is a constant matrix $\mathscr{V}_0$, indipendent from the point p $\in L_\Gamma$. Then we fix the coordinates of our manifold by parameterizing

$$p \in \mathscr{E}_\Gamma \Rightarrow p = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} \mathscr{V}_0{}^{-1}A_d\mathscr{V}_0 \\ \mathscr{V}_0{}^{-1}B_d\mathscr{V}_0 \\ \mathscr{V}_0{}^{-1}C_d\mathscr{V}_0 \end{pmatrix} \text{ where } \begin{pmatrix} A_d \\ B_d \\ C_d \end{pmatrix} \in L_\Gamma \tag{8.10}$$

It follows that equation (8.8) can be substituted by

$$\mathbb{V}_{|\Gamma|+2} \equiv \mathscr{D}_\Gamma = \mathrm{Orbit}_{\mathscr{G}_\Gamma}(\mathscr{E}_\Gamma) \tag{8.11}$$

We can also introduce a subspace $\mathscr{D}_\Gamma{}^0 \subset \mathbb{V}_{|\Gamma|+2}$ which is the orbit of $\mathscr{E}_\Gamma$ under the compact subgroup $\mathscr{F}_\Gamma$:

$$\mathscr{D}_\Gamma{}^0 = \mathrm{Orbit}_{\mathscr{F}_\Gamma}(\mathscr{E}_\Gamma) \tag{8.12}$$

This being the case we consider the restriction of the moment map to $\mathscr{D}_\Gamma$

$$\mu : \mathscr{D}_\Gamma \longrightarrow \mathbb{F}_\Gamma{}^* \tag{8.13}$$

and given an element

$$\zeta \in \mathfrak{z}[\mathbb{F}_\Gamma]^* \tag{8.14}$$

we define:

$$\mathscr{N}_\zeta \equiv \mu^{-1}(\zeta) \subset \mathscr{D}_\Gamma \quad : \quad \mathscr{N}_\zeta = \{p \in \mathscr{D}_\Gamma | \mu(p) = \zeta\} \tag{8.15}$$

Obviously we have:

$$\mathscr{N}_0 \equiv \mu^{-1}(0) = \mathscr{D}_\Gamma{}^0 \tag{8.16}$$

8.2 The space $\mathscr{N}_\zeta$ as a principal fiber bundle

The space $\mathscr{N}_\zeta$ has a natural structure of an $\mathscr{F}_\Gamma$ principal line bundle over the quotient $\mathscr{M}_\zeta$:

$$\mathscr{N}_\zeta \xrightarrow{\pi} \mathscr{M}_\zeta \equiv \mathscr{N}_\zeta // \mathscr{F}_\Gamma \tag{8.17}$$

On the tangent space to the total space of the $\mathscr{F}_\Gamma$–bundle T$\mathscr{N}_\zeta$ we have a metric induced, as the pullback, by the inclusion map:

$$\iota : \mathscr{N}_\zeta \longrightarrow \mathscr{S}_\Gamma \tag{8.18}$$

of the flat metric g on $\mathscr{S}_\Gamma$

$$g_{\mathscr{N}} = \iota^*(g_{\mathscr{S}_\Gamma}) \tag{8.19}$$

Since the metric $g_{\mathscr{S}_\Gamma}$ is Kählerian we have a Kähler potential $\mathscr{K}_{\mathscr{S}_\Gamma}$ from which it derives and we define the function

$$\mathscr{K}_{\mathscr{N}} \equiv \iota^*(\mathscr{K}_{\mathscr{S}_\Gamma}) \tag{8.20}$$

This function is not the Kähler potential of $\mathscr{N}_\zeta$ which is not even Kählerian (it has odd dimensions) but it will be related to the Kähler potential of the final quotient $\mathscr{M}_\zeta$ by means of an argument due to [42], that we spell out a few lines below. Let us denote by $p \in \mathscr{M}_\zeta$ a point of the base manifold and by $\pi^{-1}(p)$ the $\mathscr{F}_\Gamma$-fiber over that point.

8.2.1 The natural connection and the tautological bundles

We can determine a natural connection on the principal bundle (8.17) through the following steps. As it is observed in eq. (2.7) of the paper by Degeratu and Walpuski [2], which agrees with the formulae of the present paper, the quiver group has always the following form:

$$\mathscr{G}_\Gamma = \prod_{i=1}^{r} \mathrm{GL}\left(\mathbb{C}^{\dim[\boldsymbol{D}_i]}\right) \tag{8.21}$$

where $\boldsymbol{D}_i$ are the nontrivial irreducible representations of the finite group Γ, with the esclusion of $\boldsymbol{D}_0$, the identity representation. It also follows that the compact gauge subgroup $\mathscr{F}_\Gamma$ has the corresponding following structure

$$\mathscr{F}_\Gamma = \prod_{i=1}^{r} \mathrm{U}(\dim[D_i]) \tag{8.22}$$

Consequently, the principal bundle (8.17) induces holomorphic vector bundles of rank $\dim[\boldsymbol{D}_i]$ on which the compact structural group acts non-trivially only with its component $\mathrm{U}(\dim[\boldsymbol{D}_i])$. A natural connection on these bundles is obtained as it follows

$$\mathbb{A} = \frac{\mathrm{i}}{2}\left(\mathscr{H}^{-1}\partial\mathscr{H} - \mathscr{H}\overline{\partial}\mathscr{H}^{-1}\right) + g^{-1}dg \in \overset{r}{\underset{i=1}{\oplus}} \mathbf{u}(\dim[D_i]) \tag{8.23}$$

where $\mathscr{H}$ is a hermitian fiber–metric on the direct sum of the tautological vector bundles defined below:

$$\mathscr{R} \equiv \bigoplus_{i=1}^{r} \mathscr{R}_i \quad ; \quad \mathscr{R}_i \stackrel{\pi}{\longrightarrow} \mathscr{M}_\zeta \quad ; \quad \forall p \in \mathscr{M}_\zeta \quad : \quad \pi^{-1}(p) \simeq \mathbb{C}^{\dim[D_i]} \tag{8.24}$$

By definition the matrix $\mathscr{H}$ must be of dimension

$$\dim[\mathscr{H}] = n \times n \qquad \text{where} \qquad n = \sum_{i=1}^{r} \dim[D_i] = \sum_{i=1}^{r} n_i \tag{8.25}$$

In order to find the hermitian matrix $\mathscr{H}$, we argue in the following way. First we observe that in the regular representation R each irreducible representation $\boldsymbol{D}_i$ is contained exactly $\dim[\boldsymbol{D}_i]$ times, so that the form of the matrix $\mathscr{V}$ corresponding to the hermitian parameterization of the coset $\frac{\mathscr{G}_\Gamma}{\mathscr{F}_\Gamma}$ has always the following form:

$$\mathscr{V} = \left(\begin{array}{|c|c|c|c|c|}
\hline
\mathfrak{H}_0 & 0 & 0 & \dots & 0 \\
\hline
0 & \mathfrak{H}_1 \otimes \mathbf{1}_{n_1 \times n_1} & 0 & \dots & \vdots \\
\hline
0 & 0 & \mathfrak{H}_2 \otimes \mathbf{1}_{n_2 \times n_2} & \dots & \vdots \\
\hline
\vdots & \dots & \dots & \dots\dots & \vdots \\
\hline
\vdots & \dots & \dots & \dots & 0 \\
\hline
0 & \dots & \dots & 0 & \mathfrak{H}_r \otimes \mathbf{1}_{n_r \times n_r} \\
\hline
\end{array}\right) \tag{8.26}$$

where n_i denotes the dimension of the i-th nontrivial representation of the discrete group Γ and from this we extract the block diagonal matrix:

$$\mathscr{H} \equiv \left(\begin{array}{|c|c|c|c|c|}
\hline
\mathfrak{H}_1 & 0 & \dots & \dots & 0 \\
\hline
0 & \mathfrak{H}_2 & \dots & \dots & \vdots \\
\hline
\vdots & \dots\dots & \dots\dots & \dots & \vdots \\
\hline
\vdots & \dots\dots & \dots & \mathfrak{H}_{r-1} & 0 \\
\hline
0 & \dots\dots & \dots & 0 & \mathfrak{H}_r \\
\hline
\end{array}\right) \tag{8.27}$$

The hermitian matrix $\mathscr{H}$ is the fiber metric on the direct sum:

$$\mathscr{R} = \bigoplus_{i=1}^{r} \mathscr{R}_i \tag{8.28}$$

of the r tautological bundles that, by construction, are holomorphic vector bundles with rank equal to the dimension of the r irreducible representations of Γ:

$$\mathscr{R}_i \xrightarrow{\pi} \mathscr{M}_\zeta \quad ; \quad \forall p \in \mathscr{M}_\zeta \quad : \quad \pi^{-1}(p) \approx \mathbb{C}^{n_i} \tag{8.29}$$

The compatible connection[7] on the holomorphic vector bundle $\mathscr{R}$ is given by:

$$\begin{aligned}
\vartheta &= \mathscr{H}^{-1} \partial \mathscr{H} = \bigoplus_{i=1}^{r} \theta_i \\
\theta_i &= \mathfrak{H}_i^{-1} \partial \mathfrak{H}_i \in \mathbb{GL}(n_i, \mathbb{C})
\end{aligned} \tag{8.30}$$

where $\mathbb{GL}(n_i, \mathbb{C})$ is the Lie algebra of $\mathrm{GL}(n_i, \mathbb{C})$ which is structural group of the i-th tautological vector bundle. The natural connection of the $\mathscr{F}_\Gamma$ principal bundle, mentioned in eq. (8.23) is just, according to a universal scheme, the imaginary part of the holomorphic connection ϑ.

8.2.2 The tautological bundles from the irrep viewpoint and the Kähler potential

From the analysis of the above section we have reached a very elegant conclusion. Once the matrix $\mathscr{V}$ is calculated as function of the level parameters ζ and of the base-manifold coordinates $(z_m, \bar{z}_m)$ $(m = 1, 2, 3)$, we

[7] Following standard mathematical nomenclature, we call compatible connection on a holomorphic vector bundle, one whose $(0, 1)$ part is the Cauchy Riemann operator of the bundle

also have the block diagonal hermitian matrix $\mathscr{H}$ which encodes the hermitian fiber metrics $\mathfrak{H}_i(\zeta, z, \bar{z})$ on each of the tautological holomorphic bundles $\mathfrak{V}_i$ whose ranks are equal, one by one, to the dimensions n_i of the irreps of Γ. The first Chern classes of these bundles are represented by the differential $(1,1)$ forms:

$$\omega_i^{(1,1)} = \frac{i}{2\pi}\,\overline{\partial}\partial \mathrm{Log}\left[\mathrm{Det}\left[\mathfrak{H}_i\right]\right] \tag{8.31}$$

Let us recall another remarkable group theoretical fact. The number r of nontrivial irreps of Γ is equal to the number r of nontrivial conjugacy classes and to the number r of generators of the center of the compact Lie algebra $\mathbb{F}_\Gamma$, hence also to the number r of level parameters ζ and to number r of holomorphic tautological bundles. Now we are in a position to derive in full generality the formula for the Kähler potential and, hence, for the Kähler metric of the resolved manifold $\mathscr{M}_\zeta$ that we anticipated in (6.56) . In view of the above discussion, we rewrite the latter as it follows:

$$\mathscr{K}_{\mathscr{M}_\zeta} = \mathscr{K}_{\mathscr{S}_\Gamma} \mid_{\mathscr{N}_\zeta} + \zeta^i \mathfrak{C}_{\mathrm{ij}} \mathrm{Log}\left[\mathrm{Det}\left[\mathfrak{H}_j\right]\right] \tag{8.32}$$

where $\mathscr{K}_{\mathscr{S}_\Gamma}$ is the Kähler potential of the flat space $\mathscr{S}_\Gamma$ and $|_{\mathscr{N}_\zeta}$ denotes its restriction to the level surface $\mathscr{N}_\zeta$, while $\mathfrak{C}_{\mathrm{IJ}}$ is an r×r constant matrix whose structure we will define and determine below. Why the matrix defined there yields the appropriate Kähler potential is what we will now explain starting from an argument introduced in 1987 by Hitchin, Karlhede, Lindström and Roček.

The HKLR differential equation and its solution Before explaining the origin of the matrix $\mathfrak{C}_{\mathrm{IJ}}$, we would like to stress that, conceptually it encodes a pairing between the level parameters (= generators of the Lie algebra center) and the tautological bundles (= irreps). If we could understand the relation between conjugacy classes with their ages and cohomology classes, then we would have a relation between irreps and conjugacy classes and we could close the three-sided relation diagram among the center $\mathfrak{z}\left[\mathbb{F}_\Gamma\right]$ and the other two items, irreps and conjugacy classes. As we are going to show, this side of the relation is based on the concept of weighted blowup. On the other hand, understanding the matrix $\mathfrak{C}_{\mathrm{IJ}}$, is a pure Lie algebra theory issue, streaming from the HKLR argument.

Hence, continuing such an argument, let us consider the flat Kähler manifold $\mathscr{S}_\Gamma$ and its Kähler potential

$$\mathscr{K} = \sum_{i=1}^{3} \mathrm{Tr}\left[A_i A_i^\dagger\right] \text{ where we have defined } A_i = \{A, B, C\} \tag{8.33}$$

The exponential of the Kähler potential is also, by definition, the hermitian metric on the Hodge line bundle:

$$\begin{array}{l} \mathscr{L}_{\mathrm{Hodge}} \xrightarrow{\pi} \mathscr{S}_\Gamma \quad \text{where} \quad \forall p \in \mathscr{S}_\Gamma \quad : \quad \pi^{-1}(p) \approx \mathbb{C}^* \\ \|W\|^2 \equiv e^{\mathscr{K}_{\mathscr{S}}} W\overline{W} \end{array} \tag{8.34}$$

Indeed, the second line of the above equation $\|W\|^2$ defines the invariant norm of any section of $\mathscr{L}_{\mathrm{Hodge}}$.

Let us know consider the action of the quiver group on $\mathscr{S}_\Gamma$ and its effect on the fiber metric h=$e^{\mathscr{K}}$. The maximal compact subgroup $\mathscr{F}_\Gamma$ is an isometry group for the Kähler metric defined by (8.33). Hence we focus on the orthogonal (with respect to the Killing form) complement of $\mathscr{F}_\Gamma$. Let

$$\mathbf{\Phi} \in \mathbb{K}_\Gamma \tag{8.35}$$

be an element of the orthogonal subspace to the maximal compact subalgebra

$$\mathbb{G}_\Gamma = \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma \tag{8.36}$$

consider the one parameter subgroup generated by this Lie algebra element

$$g(\lambda) \equiv e^{\lambda \mathbf{\Phi}} \tag{8.37}$$

The action of this group on the Kähler potential is easily calculated

$$\mathscr{K}_{\mathscr{S}}(\lambda) = \sum_{i=1}^{3} \mathrm{Tr}\left[A_i e^{2\lambda\mathbf{\Phi}} A_i^\dagger e^{-2\lambda\mathbf{\Phi}}\right] \tag{8.38}$$

Performing the derivative with respect to λ we obtain

$$\partial_\lambda \mathscr{K}_{\mathscr{S}}(\lambda)\mid_{\lambda=0} = \sum_{i=1}^{3} \mathrm{Tr}\left(\mathbf{\Phi}\left[A_i, A_i^\dagger\right]\right) \tag{8.39}$$

Then we utilize the fact that each element $\mathbf{\Phi} \in \mathbb{K}_\Gamma$ is just equal to i$\times$ $\mathbf{\Phi}_c$ where $\mathbf{\Phi}_c$ denotes an appropriate element of the compact subalgebra. Hence the above equation becomes

$$\partial_\lambda \mathscr{K}_{\mathscr{S}}(\lambda)\mid_{\lambda=0} = \mathrm{i} \times \sum_{i=1}^{3} \mathrm{Tr}\left(\mathbf{\Phi}_c\left[A_i, A_i^\dagger\right]\right) = i\mathfrak{P}_{\mathbf{\Phi}_c} \tag{8.40}$$

Let us decompose the moment map along the standard basis of compact generators. We obtain:

$$\begin{aligned}
\mathfrak{P}_{\Phi} &= \sum_{I=1}^{|\Gamma|-1} \mathbf{\Phi}^I \mathrm{Tr}\left(\mathfrak{K}_I^c\left[A_i, A_i^\dagger\right]\right) \\
&= \mathrm{i}\sum_{I=1}^{|\Gamma|-1} \mathbf{\Phi}_c^I \mathfrak{P}_I(p) = \sum_{I=1}^{|\Gamma|-1} \mathbf{\Phi}^I \mathfrak{P}_I(p) = \sum_{I=1}^{|\Gamma|-1} \mathbf{\Phi}^I \mathrm{Tr}\left(\mathfrak{K}_I\left[A_i, A_i^\dagger\right]\right)
\end{aligned} \tag{8.41}$$

where p $\in \mathscr{D}_\Gamma \subset \mathscr{S}_\Gamma$ denotes the arbitrary point in the ambient space described by the triple of matrices A_i, $\mathfrak{K}_I$=i $\mathfrak{K}_I^c$ are the $|\Gamma|$-1 noncompact generators of the quiver group $\mathscr{G}_\Gamma$ that, by construction, are just as many as the compact generators $\mathfrak{K}_I^c$ of the maximal compact subgroup $\mathscr{F}_\Gamma$. Formally integrating the above differential equation it follows that the fiber of the metric Hodge line bundle (8.34)

$$h(p) \equiv \mathrm{Exp}\left[\mathscr{K}_{\mathscr{S}}(p)\right] \tag{8.42}$$

transforms in the following way under the action of the quiver group

$$\forall g \in \mathscr{G}_\Gamma\ g : h(p) \longrightarrow h^g(p) \equiv h\left(e^{\mathrm{Log}[g]}p\right) = e^{c(g,p)}h(p) \tag{8.43}$$

where

$$\mathrm{Log}[g] \in \mathbb{G}_\Gamma \tag{8.44}$$

is an element of the quiver group Lie algebra and as such can be decomposed along a complete basis of generators

$$\mathrm{Log}[g] = \sum_{I=1}^{7} \mathbf{\Phi}^I \mathfrak{K}_I + \mathbf{\Phi}_c{}^I \mathfrak{K}^c{}_I \tag{8.45}$$

and the anomaly $c(g,p)$introduced in eq. (8.43) has, in force of the differential equation discussed above the following form:

$$c(g,p) = \sum_{I=1}^{7} \left(\mathbf{\Phi}^I + i\mathbf{\Phi}_c{}^I\right)\mathfrak{P}_I(p) \tag{8.46}$$

where $\mathfrak{P}_I(p)$ are the moment maps at point p.

Next consider the diagram

$$\mathscr{S}_\Gamma \xleftarrow{\iota} \mathscr{N}_\zeta \xrightarrow{\pi} \mathscr{M}_\zeta \equiv \mathscr{N}_\zeta/\mathscr{F}_\Gamma \tag{8.47}$$

where $\mathscr{N}_\zeta$ is the level surface and $\mathscr{M}_\zeta$ the final Kähler threefold with its associated Hodge line bundle whose curvature is the Kähler form $\mathbf{K}_\mathscr{M}$

$$\mathbf{K}_\mathscr{M} \equiv \frac{i}{2\pi}\overline{\partial}\partial\mathscr{K}_\mathscr{M} = \frac{i}{2\pi}\overline{\partial}\left(\frac{1}{h_\mathscr{M}}\partial h_\mathscr{M}\right) \tag{8.48}$$

$\mathscr{K}_\mathscr{M}$ being the Kähler potential of the resolved variety. Following HKLR, we require that

$$\pi^*\mathbf{K}_\mathscr{M} = \iota^*\mathbf{K}_{\mathscr{S}_\Gamma} \tag{8.49}$$

where $\mathbf{K}_{\mathscr{S}_\Gamma}$ is the Kähler form of the flat Kähler manifold $\mathscr{S}_\Gamma$ =Hom$_\Gamma$(Q⊗R,R). At the level of fiber metric on the associated Hodge line bundles, eq. (8.49) amounts to stating that

$$\forall p \in \mathscr{M}_\zeta \quad : \quad h_\mathscr{M}(p) = h^g_{\mathscr{S}_\Gamma}(p) = h_{\mathscr{S}_\Gamma}(g.p) = e^{c(g,p)}\, h_{\mathscr{S}_\Gamma}(p) \tag{8.50}$$

where g is an element of the quiver group that brings the point p $\in \mathscr{N}_\zeta$ on the level surface of level ζ to the reference level surface $\mathscr{N}_0$ which corresponds to the singular orbifold $\frac{\mathbb{C}^3}{\Gamma}$. Applying this to eq. (8.46) we obtain:

$$c(g,p) = \zeta^I\mathbf{\Phi}_I(p) = \zeta^I * \mathrm{Tr}\left[\mathfrak{K}_I\mathrm{Log}[g]\right] = \sum_{i=1}^{r}\zeta^I * \mathrm{Tr}\left[\mathfrak{K}_I^{\mathrm{central}}\mathrm{Log}[g]\right] \tag{8.51}$$

since the only non-vanishing levels are located in the Lie Algebra center. On the oher hand we have g = $\mathscr{H}$:

$$\mathrm{Tr}\left[\mathfrak{K}_I^{\mathrm{central}}\mathrm{Log}[\mathscr{H}]\right] \equiv \sum_{J=1}^{r}\mathfrak{C}_{\mathrm{IJ}}\mathrm{Log}\left[\mathrm{Det}\left[\mathfrak{H}_J\right]\right] \tag{8.52}$$

The above formula defines the constant matrix $\mathfrak{C}_{\mathrm{IJ}}$ and justifies the final formula (8.32). In appendix D we will calculate an example of matrix $\mathfrak{C}_{IJ}$ for a simple case of a nonabelian group Γ which leads to tautological bundles of rank larger then one. In the case of cyclic Γ the center of the Lie Algebra $\mathbb{F}_\Gamma$ coincides with the algebra itself and the matrix $\mathfrak{C}_{IJ}$ is just diagonal and essentially trivial.

Expansion to first order In the Eguchi-Hanson example, which is the only one where the explicit form of the closed forms can be derived, we have explicitly verified that the term of order one $\overset{(1)}{\omega}{}_i^{(1,1)}$ in the small ζ parameter expansion is

$$\omega_i^{(1,1)} = 0 + \sum_{n=1}^{\infty}\zeta^n\overset{(n)}{\omega}{}_i^{(1,1)} \tag{8.53}$$

cohomologous to the full form $\omega_i^{(1,1)}$. Hence it suffices to solve the moment map equations at first order in ζ (which is always possible) and we obtain a calculation of the cohomology classes of the resolved variety according to the above displayed scheme. At the same time we obtain a calculation of the Kähler potential to the very same order.

We assume that this is a general feature applying to all cases.

Dolbeault cohomology The objects we are dealing with are Dolbeault cohomology classes of the final resolved manifold $\mathscr{M}_\zeta$ which is Kähler as a result of its Kähler quotient construction.

When we say that $\omega^{p,q}$ is a harmonic representative of a nontrivial cohomology class in $H^{1,1}\left(\mathscr{M}_\zeta\right)$ we are stating that:

- The form is ∂-closed and $\overline{\partial}$-closed
$$\partial\omega^{p,q} = \overline{\partial}\omega^{p,q} = 0$$
- There do not exist forms $\phi^{p-1,q}$ and $\phi^{p,q-1}$ such that:
$$\omega^{p,q} = \partial\phi^{p-1,q} = \overline{\partial}\phi^{p,q-1}$$

The reason why the $\omega_i^{(1,1)}$ are nontrivial representatives of $(1,1)$ cohomology classes is that they are obtained as $\overline{\partial}$ of connection one-forms $\theta^{(1,0)}$ that are not globally defined. Indeed if we introduce the curvatures and the first Chern classes of the tautological vector bundles we have the elegant formula anticipated in eq. (8.31):

$$\begin{aligned} \Theta_i &= \overline{\partial}\theta_i \\ \omega_i^{(1,1)} &\equiv c_1(\mathscr{R}_i) = \mathrm{Tr}(\Theta_i) = \overline{\partial}\,\partial\log\left[\mathrm{Det}\left(\mathfrak{H}_i\right)\right] \end{aligned} \tag{8.54}$$

Comparing now with the definition of Dolbeault cohomology we see that $\omega_i^{(1,1)}$ are nontrivial cohomology classes because either

$$\theta^{(1,0)} \equiv \partial\log\left[\mathrm{Det}\left(\mathfrak{H}_i\right)\right] \quad \text{or} \quad \theta^{(0,1)} \equiv \overline{\partial}\log\left[\mathrm{Det}\left(\mathfrak{H}_i\right)\right] \tag{8.55}$$

are non-globally defined 1-forms on the base manifold. This is so because they transform nontrivially from one local trivialization of the bundle to the next one. The transition functions on the connections are determined by the transition functions on the metric $\mathscr{H}$, namely on the coset representative. Here comes the delicate point.

Where from in the Kronheimer–like construction do we know that there are different local trivializations, otherwise that the tautological bundles are nontrivial? Computationally we solve the algebraic equations for $\mathscr{H}$ in terms of the coordinates z_i $(i = 1,2,3)$ parameterizing the locus L_Γ, which is equivalent to the singular locus $\frac{\mathbb{C}^3}{\Gamma}$ and we find $\mathscr{H} = \mathscr{H}(\zeta, z)$ where ζ are the level parameters. In order to conclude that the tautological bundle is nontrivial we should divide the locus L_Γ into patches and find the transition functions of the connections θ_i from one patch to the other. Obviously the transition function must be an element of the the quiver group $g \in \mathscr{G}_\Gamma$. At the first sight it is not clear how to implement such a program, since we do not know how we should partition the locus L_Γ. Clearly the actual solution of the algebraic equations is complicated and, as we very well know, we are able to implement it only by means of a power series in ζ, yet it is obvious that this is not a case by case study. As everything else in the Kronheimer–like construction, it must be based on first principles and it is precisely these first principles that we are going to find out. It is at this level that the issue of ages is going to come into play in an algorithmic way. We begin by inspecting the only case where the final analytic form of all the construction items is available in closed form, namely the Eguchi-Hanson case.

8.3 What we see in the Eguchi-Hanson case

Let us briefly summarize what we have verified in the EH case. The space $\mathscr{N}_\zeta$ has a natural structure of principal U(1)-bundle over the quotient $\mathscr{M}_\zeta$, as the maximal compact subgroup of the quiver group $\mathscr{F}_\Gamma \subset \mathscr{G}_\Gamma$ in this case is just U(1).

$$\mathscr{N}_\zeta \xrightarrow{\pi} \mathscr{M}_\zeta \equiv \mathscr{N}_\zeta // \mathscr{F}_\Gamma \tag{8.56}$$

As $\mathscr{N}_\zeta$ is a closed submanifold of $\mathscr{S}_\Gamma$ it has an induced metric g. The vertical tangent bundle to $\mathscr{N}_\zeta$ is locally generated by the vector field

$$V_{\mathrm{vert}} = \frac{\partial}{\partial\phi} \tag{8.57}$$

Pointwise we can consider the space $\mathrm{T}\mathcal{N}_{\mathrm{hor}}$ orthogonal to the vertical tangent bundle

$$\mathrm{T}\mathcal{N}_{\mathrm{hor}} = \left\{ X \in \mathrm{T}\mathcal{N}_{\zeta} \middle| < X, \frac{\partial}{\partial \phi} > \equiv g\left(X, \frac{\partial}{\partial \phi}\right) = 0 \right\} \tag{8.58}$$

This assignment of a complement to the vertical tangent spaces is smooth and U(1)-invariant, and therefore defines a connection on the principal bundle $\mathcal{N}_{\zeta}$, whose connection form $\boldsymbol{A}$ satisfies

$$\forall X \in \mathrm{T}\mathcal{N}_{\mathrm{hor}} : \boldsymbol{A}(X) = 0 \; ; \boldsymbol{A}\left(\frac{\partial}{\partial \phi}\right) = 1 \tag{8.59}$$

In the chosen coordinates we find:

$$\boldsymbol{A} = \mathrm{d}\phi - \frac{\zeta \mathrm{d}\theta_1 \rho_1^2}{2\left(1+\rho_1^2\right)\sqrt{\zeta^2 + 64\left(1+\rho_1^2\right)^2 \rho_2^4}} - \frac{\zeta \mathrm{d}\theta_2}{2\sqrt{\zeta^2 + 64\left(1+\rho_1^2\right)^2 \rho_2^4}} \tag{8.60}$$

where:

$$z_{1,2} = \exp\left[\mathrm{i}\,\theta_{1,2}\right] \rho_{1,2} \tag{8.61}$$

are the standard complex coordinates labeling the points of the locus L_{Γ}, namely parametrizing the two matrices A, B that solve the invariance constraint of Γ, defining $\mathrm{Hom}_{\Gamma}(\mathcal{Q} \times R, R)$, and are also diagonal in the natural basis of the regular representation. In the split basis they turn out to be antidiagonal:

$$A = \begin{pmatrix} 0 & Z_1 \\ Z_1 & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & Z_2 \\ Z_2 & 0 \end{pmatrix} \tag{8.62}$$

By means of the usual correspondence between $\mathrm{U}(1)$ bundles and line bundles we conclude that this connection $\boldsymbol{A}$ is the imaginary part of the connection theta of the corresponding bundle and we write the equation:

$$\theta = \mathfrak{H}^{-1} \partial \mathfrak{H} \tag{8.63}$$

where the explicit solution of the algebraic moment map equations yields:

$$\mathfrak{H} = \frac{\sqrt[4]{\frac{\zeta + \sqrt{\zeta^2 + 16\left|Z_1^2\right|^2 + \left|Z_2^2\right|^2}}{|Z_1|^2 + |Z_2|^2}}}{\sqrt{2}} \tag{8.64}$$

Curvature of the line bundle In this way we find that the tautological bundle has the following curvature:

$$\Theta = \overline{\partial}\partial \mathrm{Log}[\mathfrak{H}] \tag{8.65}$$

Θ is the first Chern class of the tautological line bundle implicitly defined by the above construction

$$\begin{aligned} &\mathcal{L} \xrightarrow{\pi} \mathcal{M}_{\zeta} \\ &c_1(\mathcal{L}) = \left[\frac{i}{2\pi}\Theta\right] \in H^{1,1}\left(\mathcal{M}_{\zeta}\right) \end{aligned} \tag{8.66}$$

where $H^{1,1}\left(\mathscr{M}_\zeta\right)$ is the (1,1) cohomology group of the manifold $\mathscr{M}_\zeta$. On the other hand the very space of Eguchi-Hanson $\mathscr{M}_\zeta$ is a line bundle over $\mathbb{P}_1$:

$$\mathscr{M}_\zeta \xrightarrow{\pi_0} \mathbb{P}_1 \tag{8.67}$$

There is a (1,1)-form ω over $\mathbb{P}_1$ which is the the first Chern class of the bundle $\mathscr{M}_\zeta$.

$$c_1\left(\mathscr{M}_\zeta\right) = \omega \in H^{1,1}\left(\mathbb{P}_1\right) \tag{8.68}$$

We find that, as usual the pullback $\pi_0{}^*$of the projection π_0 works in particular as follows:

$$\pi_0{}^* : \; T_{(1,1)}{}^*\mathbb{P}_1 \longrightarrow T_{(1,1)}{}^*\mathscr{M}_\xi \tag{8.69}$$

We find that the (1,1)-form Θ which is defined over the whole $\mathscr{M}_\zeta$ is the pullback image of the first Chern class of the line bundle $\mathscr{M}_\zeta$.

$$\pi_0{}^*\left[c_1\left(\mathscr{M}_\zeta\right)\right] \; = \; c_1(\mathscr{L}) \tag{8.70}$$

The line bundle $\mathscr{M}_\zeta \xrightarrow{\pi_0} \mathbb{P}_1$ is by definition the one associated with the vanishing locus of the section ξ_2.

What we have learned from this explicit case? The above detailed analysis reveals that, according to general lore, the cohomology classes constructed as first Chern classes of the tautological holomorphic vector bundles defined by the Kähler quotient via hermitian matrices $\mathfrak{H}_i$, are naturally associated with the components of the exceptional divisor. This latter is defined as the vanishing locus of a global holomorphic section $W(p)$ of a line bundle:

$$\begin{array}{l} \mathscr{L}_{\mathfrak{D}} \xrightarrow{\pi} \mathscr{M}_\zeta \\ \mathfrak{D} \subset \mathscr{M}_\zeta \quad ; \quad \mathfrak{D} = \left\{p \in \mathscr{M}_\zeta \mid W(p) = 0 \quad \text{where} \quad W \in \Gamma\left(\mathscr{L}_{\mathfrak{D}}\right)\right\} \end{array} \tag{8.71}$$

The line bundle $\mathscr{L}_{\mathfrak{D}}$ is singled out by the divisor $\mathfrak{D}$ and for this reason it is labeled by it. Its first Chern class $\omega_{\mathfrak{D}}^{(1,1)}$ is certainly a cohomology class and so it must be a linear combination of the first Chern classes $\omega_i^{(1,1)}$ created by the Kähler quotient and associated with the hermitian matrices $\mathfrak{H}_i(\zeta, p)$:

$$\left[\omega_{\mathfrak{D}}^{(1,1)}\right] = \mathrm{S}_{\mathfrak{D},i}\left[\omega_i^{(1,1)}\right] \tag{8.72}$$

The question is to know which is which and to determine the constant matrix $\mathrm{S}_{\mathfrak{D},i}$.

Another point revealed by the analysis of the Eguchi-Hanson case is that, at least locally, the entire space $\mathscr{M}_\zeta$ can be viewed as the total space of a line bundle over the divisor $\mathfrak{D}$:

$$\begin{array}{l} \mathscr{M}_\zeta \xrightarrow{\pi_d} \mathfrak{D} \\ \forall p \in \mathfrak{D} \quad ; \quad \pi_d^{-1}(p) \simeq \mathbb{C}^\star \end{array} \tag{8.73}$$

Furthermore the matrix $\mathfrak{H}_i$ can be viewed as the invariant norm of a section of the appropriate line bundle:

$$\mathfrak{H}_i(\zeta, z, \bar{z}) = H_i(\xi, \overline{\xi}, W, \overline{W})\, |W|^2 \tag{8.74}$$

where ξ denote the two coordinates spanning the divisor $\mathfrak{D}$ and W (as in fig.3) spans the vertical fibers out of the divisor. The projection π_d corresponds to setting $W \to 0$ and obtaining:

$$\pi_d : H(\xi, \overline{\xi}, W, \overline{W}) \longrightarrow h(\xi, \overline{\xi}) \equiv H(\xi, \overline{\xi}, 0, 0) \tag{8.75}$$

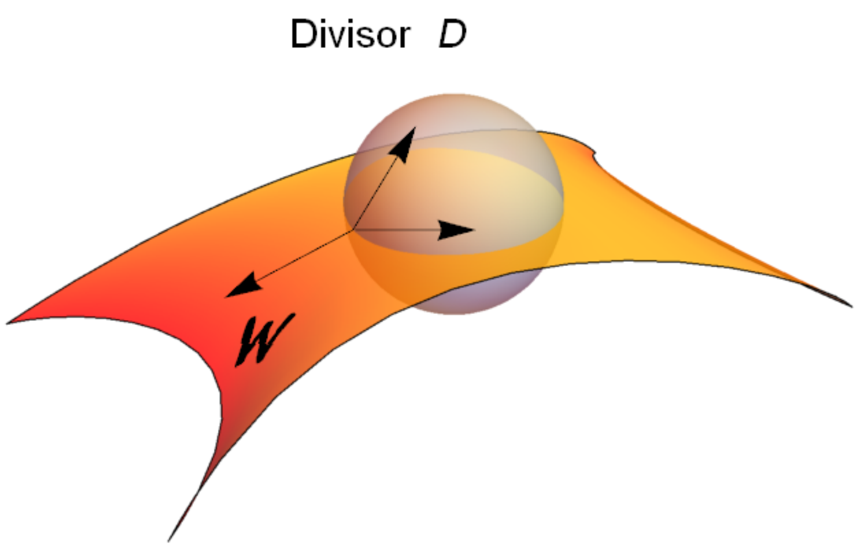


Figure 3: In the Eguchi-Hanson case the exceptional divisor is a submanifold $\mathfrak{D} \subset \mathscr{M}_\zeta$ of codimension one that is mapped into the singular point by the resolving morphism $\mathscr{M}_\zeta \longrightarrow \frac{\mathbb{C}^2}{\Gamma}$. There is a projection operation $\mathscr{M}_\zeta \xrightarrow{\pi} \mathfrak{D}$ that makes $\mathscr{M}_\zeta$ the total space of a line bundle over the divisor. Accordingly we can choose a local coordinate system for $\mathscr{M}_\zeta$ such that two coordinates span the divisor while the third, named W, transforms as if it were a section of the mentioned line bundle.

Just as in the case of Eguchi-Hanson, we expect that the two (1,1)-forms:

$$\begin{aligned} \Omega_i &= \overline{\partial}\partial H_i(\xi,\overline{\xi},W,\overline{W}) \\ \widehat{\Omega}_i &= \overline{\partial}\partial h(\xi,\overline{\xi}) \end{aligned} \tag{8.76}$$

should be cohomologous:

$$[\Omega_i] = \left[\widehat{\Omega}_i\right] \tag{8.77}$$

The form $\widehat{\Omega}_i$ is the first Chern class of the line bundle (8.73) while Ω_i is the first Chern class of the line bundle (8.71) that defines the divisor.

Divisors and conjugacy classes graded by age. Hence the question boils down to the following: *What are the components of the exceptional divisor of a crepant resolution of the singularity $\mathbb{C}^3/\Gamma$, and how many are they?* The answer is provided by Theorem 4.1 (Theorem 1.6 in [36]); they are the inverse images via the blowdown morphism of the irreducible components of the fixed locus of the action of Γ on $\mathbb{C}^3$, and are in a one-to-one correspondence with the junior conjugacy classes of Γ. The irreducible components of the exceptional divisor may be compact (corresponding to a component of the fixed locus which is just the origin of $\mathbb{C}^3$) or noncompact (corresponding to fixed loci of higher dimensions, i.e., curves).

Let us consider the case of a cyclic group Γ, with only the origin as fixed locus, and choose a generator γ of Γ of order r. As in eq. (4.7), we can write $\gamma = \frac{1}{r}(a_1, a_2, a_3)$. As described in [36], Sections 2.3 and 2.4, the resolution of singularities is obtained by iterating the following construction, which uses toric geometry (a general reference for toric geometry, which in particular explains how to perform a toric blowup by subdividing the fan of the toric variety one wants to blowup, is [46]). The fan of the toric variety $\mathbb{C}^3$ is the first octant of $\mathbb{R}^3$, with all its faces; by adding the ray $\frac{1}{r}(a_1, a_2, a_3)$ we perform a blowup $\mathbb{B}_{[a_1,a_2,a_3]} \to \mathbb{C}^3$ whose exceptional divisor F is the weighted projective space $\mathbb{WP}[a_1, a_2, a_3]$. The same procedure applied to $\mathbb{C}^3/\Gamma$ produces a partial desingularization $W_\gamma \to \mathbb{C}^3/\Gamma$ which is the base of a cyclic covering $\mathbb{B}_{[a_1,a_2,a_3]} \to W_\gamma$, ramified along the exceptional divisor E of $W_\gamma \to \mathbb{C}^3/\Gamma$. The situation is summarised by the following diagram

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & \mathbb{B}_{[a_1,a_2,a_3]} \arrow[r, "\text{weighted blowup}"] \arrow[d] & \mathbb{C}^3 \arrow[d] \\ E \arrow[r, hook] & W_\gamma \arrow[r, "\text{crepant resolution}"] & \mathbb{C}^3/\Gamma \end{tikzcd} \quad . \tag{8.78}$$

The full desingularization is obtained by performing a multiple toric blowup, adding all rays corresponding to junior conjugacy classes.

8.4 Steps of a weighted blowup

8.4.1 Weighted projective planes

Let us define in a pedantic way the weighted blowup of the origin in $\mathbb{C}^3$. To this effect we begin by recalling the definition of a weighted projective plane $\mathbb{WP}_{[a_1,a_2,a_3]}$, where $[a_1, a_2, a_3]$ are the weights (good references for weighted projective spaces and line bundles on them are [47–49]). We restrict our attention to the case where the weights are integers. One defines an action of $\mathbb{C}^*$ on $\mathbb{C}^3 - \{0\}$ letting

$$(y_1, y_2, y_3) \to (y_1 \lambda^{a_1}, y_2 \lambda^{a_2}, y_3 \lambda^{a_2}), \qquad \lambda \in \mathbb{C}^*.$$

The weighted projective plane $\mathbb{WP}_{[a_1,a_2,a_3]}$ is the quotient of $\mathbb{C}^3 - \{0\}$ under this action. It is by construction a complex variety of dimension 2.

To examine the properties of this space it is expedient to assume that the triple (a_1, a_2, a_3) is reduced. One defines

$$d_i = \text{g.c.d.}\,(a_{i-1}, a_{i+1}), \qquad b_i = \text{l.c.m.}(d_{i-1}, d_{i+1})$$

(where indices in the r.h.s. are meant mod 3, i.e., $1-1=3$, etc.). The triple (a_1, a_2, a_3) is reduced if $(b_1, b_2, b_3) = (1, 1, 1)$; otherwise one defines $a_i' = a_i/b_i$. The numbers a_i' are positive integers, the triple (a_1', a_2', a_3') is reduced, and the weighted projective planes $\mathbb{WP}_{[a_1,a_2,a_3]}$ and $\mathbb{WP}_{[a_1',a_2',a_3']}$ are isomorphic. Henceforth we shall assume that the triple (a_1, a_2, a_3) is reduced. It turns out that $\mathbb{WP}_{[a_1,a_2,a_3]}$ is smooth if and only if $(a_1, a_2, a_3) = (1, 1, 1)$, in which case the weighted projective plane is just $\mathbb{P}^2$.

The same construction of line bundles on projective spaces (see e.g. [50], Section II.5) produces on weighted projective spaces rank one sheaves $\mathscr{O}_{\mathbb{WP}_{[a_1,a_2,a_3]}}(i)$, with $i \in \mathbb{Z}$, that in general are not locally free (i.e., they are not line bundles), but only reflexive (i.e., they are isomorphic to their duals). It turns out that $\mathscr{O}_{\mathbb{WP}_{[a_1,a_2,a_3]}}(i)$ is locally free if and only if i is a multiple of $m = \text{l.c.m.}(a_1, a_2, a_3)$ [48].

The weighted projective plane $\mathbb{WP}_{[a_1,a_2,a_3]}$ is covered by the open sets

$$U_i = \{(y_1, y_2, y_3) \,|\, y_i \neq 0\}.$$

On this open cover the line bundle $\mathscr{O}_{\mathbb{WP}_{[a_1,a_2,a_3]}}(km)$ has transition functions

$$g_{ij}\colon U_i \cap U_j \to \mathbb{C}^*, \qquad g_{ij}(y_1,y_2,y_3) = y_j^{km/a_j} y_i^{-km/a_i} \tag{8.79}$$

where $m = \text{l.c.m.}\ (a_1,a_2,a_3)$. In particular, (the isomorphism class of) $\mathscr{O}_{\mathbb{WP}_{[a_1,a_2,a_3]}}(m)$ is the (very ample) generator of the Picard group of $\mathbb{WP}_{[a_1,a_2,a_3]}$, the group of isomorphism classes of line bundles on $\mathbb{WP}_{[a_1,a_2,a_3]}$, which is isomorphic to $\mathbb{Z}$. We conclude this brief introduction to weighted projective planes by defining an orbifold Kähler metric for the spaces $\mathbb{WP}_{[a_1,a_2,a_3]}$. Denoting again by (y_1,y_2,y_3) a set of homogeneous coordinates on $\mathbb{WP}_{[a_1,a_2,a_3]}$, and $m = \text{l.c.m.}\,(a_1,a_2,a_3)$, one can check the 2-form

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log\sum_{i=1}^{3} y_i^{m/a_i}\,\overline{y}_i^{m/a_i}$$

is invariant under rescaling of the homogeneous coordinates, and therefore defines a 2-form on the smooth locus of $\mathbb{WP}_{[a_1,a_2,a_3]}$; this reduces to the usual Fubini-Study form when the projective space is smooth.

8.4.2 The weighted blowup and the tautological bundle

The weighted blowup of $\mathbb{C}^3$, denoted $\mathbb{B}_{[a_1,a_2,a_3]}$, is a subvariety

$$\mathbb{B}_{[a_1,a_2,a_3]} \subset \mathbb{C}^3 \times \mathbb{WP}_{[a_1,a_2,a_3]} \tag{8.80}$$

defined by the equations

$$z_1 y_2^{a_1 a_3} = z_2 y_1^{a_2 a_3}, \qquad z_2 y_3^{a_1 a_2} = z_3 y_2^{a_1 a_3}, \qquad z_1 y_3^{a_1 a_2} = z_3 y_1^{a_2 a_3} \tag{8.81}$$

where $\{z_1,z_2,z_3\}$ are standard coordinates in $\mathbb{C}^3$, and $\{y_1,y_2,y_3\}$ are homogenous cooordinates in $\mathbb{WP}_{[a_1,a_2,a_3]}$. Actually the three equations are not independent (regarding them as a linear system in the unknowns z, the associated matrix has rank at most 2) and therefore the locus $\mathbb{B}_{[a_1,a_2,a_3]}$ is 3-dimensional. The projections of $\mathbb{C}^3 \times \mathbb{WP}_{[a_1,a_2,a_3]}$ onto its factors define projections

$$\begin{array}{rcl} p & : & \mathbb{B}_{[a_1,a_2,a_3]} \to \mathbb{C}^3 \\ \pi & : & \mathbb{B}_{[a_1,a_2,a_3]} \to \mathbb{WP}_{[a_1,a_2,a_3]} \end{array}$$

From eq. (8.81) we see that the fibers of π are isomorphic to $\mathbb{C}$; indeed, by comparing with eq. (8.79), we see that $\mathbb{B}_{[a_1,a_2,a_3]}$ is the total space of the line bundle $\mathscr{O}_{\mathbb{WP}_{[a_1,a_2,a_3]}}(-1)$ over the base $\mathbb{WP}_{[a_1,a_2,a_3]}$, and π is the bundle projection. On the other hand, the morphism p is birational, as it is an isomorphism away from the fiber $p^{-1}(0)$, while the fiber itself — the exceptional divisor F of the blowup — is isomorphic to $\mathbb{WP}_{[a_1,a_2,a_3]}$.

The blowup $\mathbb{B}_{[a_1,a_2,a_3]}$ is nicely described in terms of toric geometry [46]. Denoting by $\{e_i\}$ the standard basis of $\mathbb{R}^3$, the variety $\mathbb{B}_{[a_1,a_2,a_3]}$ is associated with the fan given by the one-dimensional cones (rays) generated by

$$e_1, \qquad e_2, \qquad e_3, \qquad v = a_1 e_1 + a_2 e_2 + a_3 e_3.$$

The fan has three 3-dimensional cones σ_i, corresponding to 3 open affine toric varieties U_i which cover $\mathbb{B}_{[a_1,a_2,a_3]}$ (see Figure 4). It turns out that U_i is smooth if and only if $a_i = 1$, so that $\mathbb{B}_{[a_1,a_2,a_3]}$ is smooth if and only if $a_1 = a_2 = a_3 = 1$ (in which case the exceptional divisor is a $\mathbb{P}^2$). Moreover, unless again $a_1 = a_2 = a_3 = 1$, F is a Weil divisor, so that its associated rank one sheaf (it ideal sheaf, i.e., the sheaf of functions $\mathbb{B}_{[a_1,a_2,a_3]}$ that vanish on F), is not locally free, but only reflexive. We shall denote the dual of this sheaf as $\mathscr{O}_{\mathbb{B}_{[a_1,a_2,a_3]}}(F)$. Although this in general is not locally free, it is still true that its first Chern class if Poincaré dual to F.

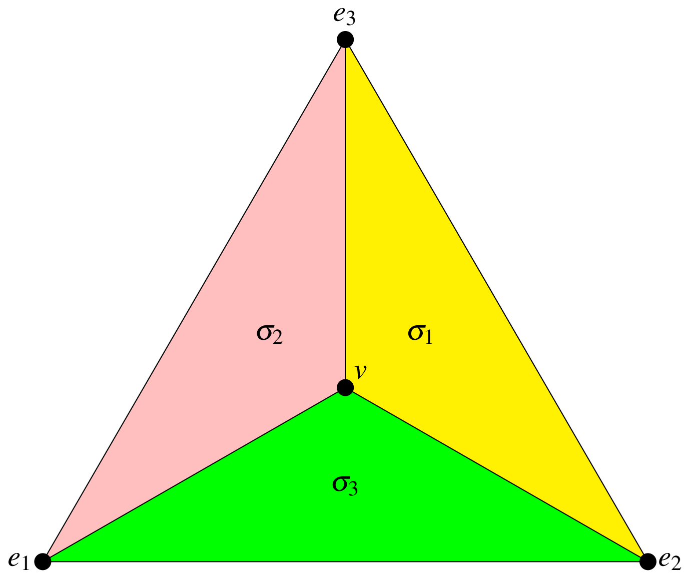


Figure 4: Representation of the fan of $\mathbb{B}_{[a_1,a_2,a_3]}$. The vector v *is not* on the plane singled out by e_1, e_2, e_3.

Let us assume that we are interested in desingularizing a quotient $\mathbb{C}^3/\Gamma$, where Γ is cyclic, and the representation of Γ on $\mathbb{C}^3$ has just one junior class $\frac{1}{r}(a_1,a_2,a_3)$, which is compact, i.e., all a_i are strictly positive. The geometric constructions in this section show that the orbifold Kähler form on $\mathbb{WP}_{[a_1,a_2,a_3]}$ induce by pullback a Kähler form on the smooth locus of the blowup $\mathbb{B}_{[a_1,a_2,a_3]}$. With reference to diagram (8.78), the action of the group Γ leaves the exceptional divisor F pointwise fixed, so that the form descends to the desingularization $\mathscr{M}_\zeta$ of $\mathbb{C}^3/\Gamma$. However, this Kähler form does not appear to coincide with the Kähler form built on $\mathscr{M}_\zeta$ by means of the Kähler reduction (cf. Section D). This comes to no surprise as it is known that there are several different metrics on a weighted projective space which all reduce to the standard Fubini-Study metric in the smooth case (see [51]). We shall analyze the relation between the naive metric introduced above and the one obtain by Kähler reduction in a future work.

8.4.3 Pairing between irreps and conjugacy class in the Kähler quotient resolution: open questions

According to [36], in the crepant resolution:

$$\mathscr{M}_\zeta \longrightarrow \frac{\mathbb{C}^3}{\Gamma} \tag{8.82}$$

we obtain a component of the exceptional divisor $\mathfrak{D}^{(E)}$ for each junior conjugacy class of the group Γ, namely we have:

$$\mathfrak{D}^{(E)} = \bigcup_{i=1}^{\#\text{ of junior classes}} \mathfrak{D}_{[a_1,a_2,a_3]_i} \tag{8.83}$$

When there is just one junior class, the procedure described in previous subsections, which is graphically summarized in the diagram (8.78), is exhaustive and we easily identify the exceptional divisor with a single projective plane $\mathbb{WP}_{[a_1,a_2,a_3]}$. Indeed, the divisor F is the weighted projective plane $\mathbb{WP}_{[a_1,a_2,a_3]}$ by construction, and the action of Γ leaves it pointwise fixed, so that E is isomorphic to F.

Utilizing the correspondence between line bundles and divisors, we can conclude that the exceptional divisor $\mathbb{WP}_{[a_1,a_2,a_3]}$ uniquely identifies a line-bundle, *i.e.*, the tautological line bundle $\mathscr{T}_{[a_1,a_2,a_3]}$, whose first Chern class is necessarily given by:

$$c_1\left(\mathscr{T}_{[a_1,a_2,a_3]}\right) = \frac{\mathrm{i}}{2\pi}\overline{\partial}\,\partial\,\log\left(H_{[a_1,a_2,a_3]}\right) \tag{8.84}$$

where $H_{[a_1,a_2,a_3]}$ is a suitable hermitian fiber-metric. The most interesting issue is to relate such an invariant fiber metric and the (1,1)-form $c_1\left(\mathscr{T}_{[a_1,a_2,a_3]}\right)$ with the real functions $\mathfrak{H}_i$ in eq.(8.27) and the corresponding (1,1)-forms (8.31), that are produced by the Kähler quotient construction.

In the next section such a relation will be explicitly analyzed in the context of a simple master example that indeed is characterized by a unique *junior conjugacy class*.

The construction of the exceptional divisor and the structure of the blowup in cases with several junior classes is more complicated and it is still under investigation. The general pairing rules between irreps and conjugacy classes will be discussed and elucidated in a future publication by the present authors. In the next section we briefly outline the general problem before presenting our explicit results for the above mentioned one junior class master model.

9 Analysis of the (1,1)-forms: irreps versus conjugacy classes that is cohomology versus homology

In the present section we plan to analyze in full detail, within the scope of a one junior class model, the relation between the above extensively discussed $\omega_\alpha^{(1,1)}$ forms ($\alpha = 1,\dots,r = \#$ of nontrivial irreps), with the exceptional divisors generated by the blowup of the singularity, together with the other predictions of the fundamental theorem 4.1 which associates cohomology classes of $\mathscr{M}_\zeta$ with conjugacy classes of Γ. The number of nontrivial conjugacy classes and the number of nontrivial irreps are equal to each other so that we use r in both cases, yet what is the actual pairing is not clear a priori and it is not intrinsic to group theory, as we have stressed several times. In this section we want to explore this pairing and to do that in an explicit way we need explicit calculable examples. These are very few because of the bottleneck constituted by the solution of the moment map equations, that are algebraic of higher degree and only seldom admit explicit analytic solutions. For this reason we introduce here the full-fledged construction of one of those rare examples, where the moment map equations are solved in terms of radicals. As anticipated above this model has the additional nice feature that the number of junior conjugacy classes is just one. It will be the master model for our explicit analysis. The construction of other examples is confined to appendices. In particular appendix D contains two Abelian examples where Γ is a cyclic group, respectively $\Gamma = \mathbb{Z}_3$ and $\Gamma = \mathbb{Z}_7$. In both cases there analyzed the chosen group Γ is a subgroup of the maximal simple group $\mathrm{L}_{168} \simeq \mathrm{PSL}(2,7)$ whose action on $\mathbb{C}^3$ was considered by Markushevich in [32]. The master model that we discuss here is also based on $\Gamma = \mathbb{Z}_3$ but the action of the latter on $\mathbb{C}^3$ is differently constructed. The radically different results of appendix D.1 from those of the next section 9.1 should advise the reader of the relevance of the embedding:

$$\iota \quad : \quad \Gamma \hookrightarrow \mathrm{SU}(3) \tag{9.1}$$

In appendix E we also provide the construction of the simplest possible nonabelian model that corresponds to the choice $Dih_3 \sim S_3$. Also for this toy model the moment maps are algebraic of higher degree and an analytic solution is out of question. Nevertheless the nonabelian cases are our main final target and we will return to them in future publications.

It is also important to stress that aim of the Kronheimer-like construction is not only the calculation of cohomology but also the actual determination of the Kähler potential (yielding the Kähler metric), which is encoded in formula (8.32). From this point of view one of the $\mathrm{Det}\mathfrak{H}_i$ may lead to a corresponding $\omega_i^{(1,1)} = \frac{i}{2\pi}\overline{\partial}\partial\mathrm{Det}\mathfrak{H}_i$ that is either exact or cohomologous to another one, yet its contribution to the Kähler potential, which is very important in physical applications, can not be neglected. It is only the cohomology class of the

Kähler 2-form that is affected by the triviality of one or more of the $\omega_i^{(1,1)}$; the contributions to the Kähler potential that give rise to exact form deformations of the Kähler 2-form are equally important as others.

Having anticipated these general considerations we turn to our master model.

9.1 The master model $\frac{\mathbb{C}^3}{\Gamma}$ with generator $\{\xi,\xi,\xi\}$

In this section we develop all the calculations for the Kähler quotient resolution of the quotient singularity $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$ in the case where the generator Y of $\mathbb{Z}_3$ is of the following form:

$$Y = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix} \tag{9.2}$$

ξ being a primite cubic root of unity ξ^3 =1.
The equation $p \wedge p$=0 which is a set of quadrics has solutions arranged in various branches. There is a unique, principal branch of the solution that has maximal dimension $\mathscr{D}_\Gamma^0$ and is indeed isomorphic to the $\mathscr{G}_\Gamma$ orbit of the singular locus L_Γ . This principal branch is the algebraic variety $\mathbb{V}_{|\Gamma|+2}$ mentioned in eq. (3.40), of which we perform the Kähler quotient with respect to the group $\mathscr{F}_\Gamma$

$$\mathscr{F}_\Gamma = \bigotimes_{\mu=1}^{r+1} \mathrm{U}\left(n_\mu\right) \cap \mathrm{SU}(|\Gamma|) = \mathrm{U}(1) \otimes \mathrm{U}(1) \tag{9.3}$$

in order to obtain the crepant resolution together with its Kähler metric. In the above formula n_μ = $\{1,1,1\}$ are the dimensions of the irreducible representations of Γ=$\mathbb{Z}_3$ and r+1=3 is the number of conjugacy classes of the group (r is the number of nontrivial representations).
To make a long story short, exactly as in the Kronheimer case we are able to retrieve the algebraic equation of the singular locus from traces and determinants of the quiver matrices restricted to L_Γ. Precisely for the $\mathbb{Z}_3$ case under consideration we obtain

$$\mathscr{I}_1 = \mathrm{Det}\left[A_o\right]; \mathscr{I}_2 = \mathrm{Det}\left[B_o\right];\ \mathscr{I}_3 = \mathrm{Det}\left[C_o\right]; \mathscr{I}_4 = \frac{1}{3}\mathrm{Tr}\left[A_o B_o C_o\right] \tag{9.4}$$

and we find the relation

$$\mathscr{I}_1 \mathscr{I}_2 \mathscr{I}_3 = \mathscr{I}_4{}^3 \tag{9.5}$$

which reproduces the $\mathbb{C}^3$ analogue of eqs. (5.37-5.39) applying to the $\mathbb{C}^2$ case of Kronheimer and Arnold.

The main difference, as we have several time observed, is that now the same eqs. remain true, with no deformation for the entire $\mathscr{G}_\Gamma$ = $\mathbb{C}^* \times \mathbb{C}^*$, orbit of the locus L_Γ, namely for the entire $\mathbb{V}_{|\Gamma|+2}$ = $\mathbb{V}_5$ variety of which we construct the Kähler quotient with respect to the compact subgroup $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathbb{C}^* \times \mathbb{C}^*$. This is in line with the many times emphasized feature that in the $\mathbb{C}^3$ case there is no deformation of the complex structure.

9.1.1 The actual calculation of the Kähler quotient and of the Kähler potential

The calculation of the final form of the Kähler potential is reduced to the solution of a set of two algebraic equations. The solutions of such equations are accessible in this particular case since they reduce to a single cubic for which we have Cardano's formula. For this reason the present case is the three-dimensional analogue of the Eguchi-Hanson space where everything is explicitly calculable and all theorems admit explicit testing and illustration.

By calculating the ages we determine the number of $\omega^{(q,q)}$ harmonic forms (where $q = 1,2$). According to theorem 4.1 all these forms (and their dual cycles in homology) should be in one-to-one correspondence with

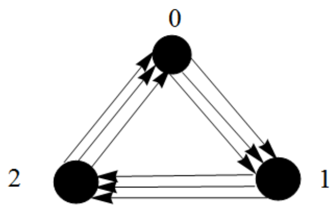


Figure 5: The quiver diagram of the cyclic group with generator $Y = \text{diag}\{\xi, \xi.\xi\}$.

the r nontrivial conjugacy classes of Γ. On the other hand the Kähler quotient construction associates one level parameter ζ to each generator of the center $\mathfrak{z}(\mathscr{F}_\Gamma)$ of the group $\mathscr{F}_\Gamma$, two ζ.s in this case, that are in one-to-one correspondence with the r nontrivial irreducible representation of Γ. The number is the same, but what is the pairing between **irreps** and **conjugacy classes**? More precisely how do we see the homology cycles that are created when each of the r level parameters ζ departs from its original zero value? Using the explicit expression of the functions $\mathfrak{H}_{1,2}$ defined in eqs. (8.26-8.31) we arrive at the calculation of the $\omega^{(1,1)}{}_{i=1,2}$ forms that encode the first Chern classes of the two tautological bundles. The expectation from the age argument is that these two 2-forms should be cohomologous corresponding to just the unique predicted class of type (1,1) since $h^{1,1}$=1. On the other hand we should be able to construct an $\omega^{(2,2)}$ form representing the unique class that is Poincarè dual to the exceptional divisor.

In this case we can successfully answer both questions and this is very much illuminating.

Ages. Indeed taking the explicit generator

$$Y = \begin{pmatrix} (-1)^{2/3} & 0 & 0 \\ 0 & (-1)^{2/3} & 0 \\ 0 & 0 & (-1)^{2/3} \end{pmatrix} \tag{9.6}$$

we easily calculate the $\{a_1, a_2, a_3\}$ vectors respectively associated to each of the three conjugacy classes and we obtain:

$$a - \text{vectors} = \{\{0,0,0\}, \tfrac{1}{3}\{1,1,1\}, \tfrac{1}{3}\{2,2,2\}\} \tag{9.7}$$

from which we conclude that, apart from the class of the identity, there is just one junior and one senior class.

Hence we conclude that the Hodge numbers of the resolved variety should be
$h^{(0,0)} = 1$; $h^{(1,1)} = 1$; $h^{(2,2)} = 1$.
If we follow the weighted blowup procedure described in section 8.4 using the weights of the unique junior class $\{1,1,1\}$, we see that the projection π of eq. (8.82) yields

$$\pi : \mathbb{B}_{(1,1,1)} \longrightarrow \mathbb{WP}_{(1,1,1)} \sim \mathbb{P}^2 \tag{9.8}$$

So the blowup replaces the singular point $0 \in \mathbb{C}^3$ with a $\mathbb{P}^2$, which is compact. As a result, also the expectional divisor in the resolution $\mathscr{M}_\zeta$ is compact. By Poincaré duality this entrains the existence of a harmonic (2,2)–form associated with the unique senior class.

9.1.2 The quiver matrix

In this case, the quiver matrix defined by eq. (6.1) is the following one :

$$A_{ij} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix} \tag{9.9}$$

and it has the graphical representation displayed in fig. 5

9.1.3 The space $\mathscr{S}_\Gamma = \mathrm{Hom}_\Gamma(\mathscr{Q} \otimes R, R)$ in the natural basis

Solving the invariance constraints (6.7) in the natural basis of the regular representation we find the triples of matrices $\{$A,B,C$\}$ spanning the locus $\mathscr{S}_\Gamma$. They are as follows:

$$
\begin{aligned}
A &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ (-1)^{2/3}\alpha_{1,3} & (-1)^{2/3}\alpha_{1,1} & (-1)^{2/3}\alpha_{1,2} \\ -(-1)^{1/3}\alpha_{1,2} & -(-1)^{1/3}\alpha_{1,3} & -(-1)^{1/3}\alpha_{1,1} \end{pmatrix} \\
B &= \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ (-1)^{2/3}\beta_{1,3} & (-1)^{2/3}\beta_{1,1} & (-1)^{2/3}\beta_{1,2} \\ -(-1)^{1/3}\beta_{1,2} & -(-1)^{1/3}\beta_{1,3} & -(-1)^{1/3}\beta_{1,1} \end{pmatrix} \\
C &= \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} \\ (-1)^{2/3}\gamma_{1,3} & (-1)^{2/3}\gamma_{1,1} & (-1)^{2/3}\gamma_{1,2} \\ -(-1)^{1/3}\gamma_{1,2} & -(-1)^{1/3}\gamma_{1,3} & -(-1)^{1/3}\gamma_{1,1} \end{pmatrix}
\end{aligned}
\tag{9.10}
$$

The locus L_Γ. The locus $L_\Gamma \subset \mathscr{S}_\Gamma$ is easily described by the equation:

$$
\begin{aligned}
A_0 &= \begin{pmatrix} \alpha_{1,1} & 0 & 0 \\ 0 & (-1)^{2/3}\alpha_{1,1} & 0 \\ 0 & 0 & -(-1)^{1/3}\alpha_{1,1} \end{pmatrix} \\
B_0 &= \begin{pmatrix} \beta_{1,1} & 0 & 0 \\ 0 & (-1)^{2/3}\beta_{1,1} & 0 \\ 0 & 0 & -(-1)^{1/3}\beta_{1,1} \end{pmatrix} \\
C_0 &= \begin{pmatrix} \gamma_{1,1} & 0 & 0 \\ 0 & (-1)^{2/3}\gamma_{1,1} & 0 \\ 0 & 0 & -(-1)^{1/3}\gamma_{1,1} \end{pmatrix}
\end{aligned}
\tag{9.11}
$$

9.1.4 The space $\mathscr{S}_\Gamma$ in the split basis

Solving the invariance constraints in the split basis of the regular representation we find another representation of the triples of matrices $\{A,B,C\}$ that span the space $\mathscr{S}_\Gamma$. They are as follows:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 & m_{1,3} \\ m_{2,1} & 0 & 0 \\ 0 & m_{3,2} & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 & n_{1,3} \\ n_{2,1} & 0 & 0 \\ 0 & n_{3,2} & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 0 & 0 & r_{1,3} \\ r_{2,1} & 0 & 0 \\ 0 & r_{3,2} & 0 \end{pmatrix} \end{aligned} \tag{9.12}$$

9.1.5 The equation $p \wedge p = 0$ and the characterization of the variety $\mathbb{V}_5 = \mathscr{D}_\Gamma$

Here we are concerned with the solution of eq. (6.12) and the characterization of the locus $\mathscr{D}_\Gamma$.

Differently from the more complicated cases of larger groups, in the present abelian case of small order, we can explicitly solve the quadratic equations provided by the commutator constraints and we discover that there is a principal branch of the solution, named $\mathscr{D}_\Gamma^0$ that has indeed dimension 5=$|\Gamma|$+2. In addition however there are several other branches with smaller dimension. These branches describe different components of the locus $\mathscr{D}_\Gamma$. Actually as already pointed out they are all contained in the $\mathscr{G}_\Gamma$ orbit of the subspace L_Γ defined above. The quadratic equations defining $\mathscr{D}_\Gamma$ have a set of 14 different solutions realized by a number n_i of constraints on the 9 parameters. Hence there are 14 branches $\mathscr{D}_\Gamma^i$(i=0,1,...16) of dimensions:

$$\dim_{\mathbb{C}} \mathscr{D}_\Gamma^i = 9 - n_i \tag{9.13}$$

The full dimension table of these branches is displayed below

$$\{5,4,4,4,4,3,3,3,3,3,3,3,2,2\}$$

As we see, there is a unique branch that has the maximal dimension 5 =$|\mathbb{Z}_3|+2$. This is the principal branch $\mathscr{D}_\Gamma^0$. It can be represented by the substitution:

$$n_{2,1} \to \frac{m_{2,1}n_{1,3}}{m_{1,3}}, \quad n_{3,2} \to \frac{m_{3,2}n_{1,3}}{m_{1,3}}, \quad r_{2,1} \to \frac{m_{2,1}r_{1,3}}{m_{1,3}}, \quad r_{3,2} \to \frac{m_{3,2}r_{1,3}}{m_{1,3}} \tag{9.14}$$

In this way we have reached a complete resolution of the following problem. We have an explicit parametrization of the variety $V_{|\Gamma|+2}$. This variety is described by the following three matrices depending on the 5 complex variables ω_i (i=1,...,5):

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 0 & \omega_1 \\ \omega_2 & 0 & 0 \\ 0 & \omega_3 & 0 \end{pmatrix} \\
B &= \begin{pmatrix} 0 & 0 & \omega_4 \\ \frac{\omega_2\omega_4}{\omega_1} & 0 & 0 \\ 0 & \frac{\omega_3\omega_4}{\omega_1} & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & 0 & \omega_5 \\ \frac{\omega_2\omega_5}{\omega_1} & 0 & 0 \\ 0 & \frac{\omega_3\omega_5}{\omega_1} & 0 \end{pmatrix}
\end{aligned} \tag{9.15}$$

9.1.6 The quiver group

Our next point is the derivation of the group $\mathscr{G}_\Gamma$ defined in eqs. (6.17) and (6.18), namely:

$$\mathscr{G}_\Gamma = \{g \in \mathrm{SL}(|\Gamma|,\mathbb{C}) \,|\, \forall\gamma \in \Gamma : [D_R(\gamma), D_{\mathrm{def}}(g)] = 0\} \tag{9.16}$$

Let us proceed to this construction. In the diagonal basis of the regular representation this is a very easy task, since the group is simply given by the diagonal 3×3 matrices with determinant one. We introduce such matrices

$$\mathfrak{g} \in \mathscr{G}_\Gamma \quad : \quad \mathfrak{g} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{9.17}$$

9.1.7 $\mathbb{V}_5$ as the orbit under $\mathscr{G}_\Gamma$ of the locus L_Γ

In this section we want to verify and implement eq. (3.40), namely we aim at showing that $\mathbb{V}_5$=$\mathscr{D}_\Gamma = \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$. To this effect we rewrite the locus L_Γ in the diagonal split basis of the regular representation. The change of basis is performed by the character table of the cyclic group $\mathbb{Z}_3$. The result is displayed below:

$$\begin{aligned}
A_0 &= \begin{pmatrix} 0 & 0 & \alpha_{1,1} \\ \alpha_{1,1} & 0 & 0 \\ 0 & \alpha_{1,1} & 0 \end{pmatrix} \\
B_0 &= \begin{pmatrix} 0 & 0 & \beta_{1,1} \\ \beta_{1,1} & 0 & 0 \\ 0 & \beta_{1,1} & 0 \end{pmatrix} \\
C_0 &= \begin{pmatrix} 0 & 0 & \gamma_{1,1} \\ \gamma_{1,1} & 0 & 0 \\ 0 & \gamma_{1,1} & 0 \end{pmatrix}
\end{aligned} \tag{9.18}$$

Eventually the complex parameters

$$z_1 \equiv \alpha_{1,1}; \quad z_2 \equiv \beta_{1,1}; \quad z_3 \equiv \gamma_{1,1} \tag{9.19}$$

will be utilized as complex coordinates of the resolved variety when the level parameters $\zeta_{1,2}$ are switched on. Starting from the above the orbit is given by:

$$\text{Orbit}_{\mathscr{G}_\Gamma} \equiv \left\{ \left\{ \mathfrak{g} A_0 \mathfrak{g}^{-1}, \mathfrak{g} B_0 \mathfrak{g}^{-1}, \mathfrak{g} C_0 \mathfrak{g}^{-1} \right\} \quad | \quad \forall \mathfrak{g} \in \mathscr{G}_\Gamma, \forall \{A_0, B_0, C_0\} \in L_\Gamma \right\} \supset \mathscr{D}_\Gamma^0 \tag{9.20}$$

and the correspondence between the parameters of the principal branch $\mathscr{D}_\Gamma^0$ and the parameters spanning $\mathscr{G}_\Gamma$ and L_Γ is provided below:

$$a_1 \to \frac{\omega_2^{1/3}}{\omega_1^{1/3}}, a_2 \to \frac{\omega_3^{1/3}}{\omega_2^{1/3}}, a_3 \to \frac{\omega_1^{1/3}}{\omega_3^{1/3}}, z_1 \to \omega_1^{1/3}\omega_2^{1/3}\omega_3^{1/3}, z_2 \to \frac{\omega_2^{1/3}\omega_3^{1/3}\omega_4}{\omega_1^{2/3}}, z_3 \to \frac{\omega_2^{1/3}\omega_3^{1/3}\omega_5}{\omega_1^{2/3}} \tag{9.21}$$

Branches of smaller dimension of the solution are all contained in the $\text{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$ and correspond to the orbits of special points of L_Γ where some of the z_i vanish or satisfy special relations among themselves. Hence, indeed we have:

$$\text{Orbit}_{\mathscr{G}_\Gamma} = \mathscr{D}_\Gamma$$

9.1.8 The compact gauge group $\mathscr{F}_\Gamma = \mathrm{U}(1)^2$

We introduce a basis for the generators of the compact subgroup $\mathrm{U}(1)^2 = \mathscr{F}_\Gamma \subset \mathscr{G}_\Gamma$ provided by the set of two generators displayed here below

$$T^1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \tag{9.22}$$

whose trace-normalization is the A_2 Cartan matrix

$$\text{Tr}\left(T^i T^j\right) = \mathfrak{C}^{\text{ij}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{9.23}$$

9.1.9 Calculation of the Kähler potential and of the moment maps

Naming Δ_i the moduli of the coordinates z_i and θ_i their phases according to $z_i = e^{\mathrm{i}\theta_i}\Delta_i$ and considering a generic element $\mathfrak{g}_R$ of the quiver group that is real and hence is a representative of a coset class in $\frac{\mathscr{G}_\Gamma}{\mathscr{F}_\Gamma}$:

$$\mathfrak{g}_R = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{-\lambda_1+\lambda_2} & 0 \\ 0 & 0 & e^{-\lambda_2} \end{pmatrix} \quad ; \quad \lambda_{1,2} \in \mathbb{R} \tag{9.24}$$

The triple of matrices $\{A,B,C\}=\left\{\mathfrak{g}_R A_0 \mathfrak{g}_R{}^{-1}, \mathfrak{g}_R B_0 \mathfrak{g}_R{}^{-1}, \mathfrak{g}_R C_0 \mathfrak{g}_R{}^{-1}\right\}$ have the following explicit appearance:

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & 0 & e^{i\theta_1-\lambda_1-\lambda_2}\Delta_1 \\ e^{i\theta_1+2\lambda_1-\lambda_2}\Delta_1 & 0 & 0 \\ 0 & e^{i\theta_1-\lambda_1+2\lambda_2}\Delta_1 & 0 \end{pmatrix} \\
B &= \begin{pmatrix} 0 & 0 & e^{i\theta_2-\lambda_1-\lambda_2}\Delta_2 \\ e^{i\theta_2+2\lambda_1-\lambda_2}\Delta_2 & 0 & 0 \\ 0 & e^{i\theta_2-\lambda_1+2\lambda_2}\Delta_2 & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & 0 & e^{i\theta_3-\lambda_1-\lambda_2}\Delta_3 \\ e^{i\theta_3+2\lambda_1-\lambda_2}\Delta_3 & 0 & 0 \\ 0 & e^{i\theta_3-\lambda_1+2\lambda_2}\Delta_3 & 0 \end{pmatrix}
\end{aligned}
\tag{9.25}
$$

Calculating the Kähler potential we find

$$
\mathcal{K}_{\mathcal{S}}|_{\mathcal{D}} = \left(\mathrm{Tr}\left[A\,A^\dagger\right] + \mathrm{Tr}\left[B\,B^\dagger\right] + \mathrm{Tr}\left[C\,C^\dagger\right]\right) = e^{-2(\lambda_1+\lambda_2)}\left(1+e^{6\lambda_1}+e^{6\lambda_2}\right)\left(\Delta_1^2+\Delta_2^2+\Delta_3^2\right) \tag{9.26}
$$

We have used the above notation since $\mathrm{Tr}\left[A,A^\dagger\right]+\mathrm{Tr}\left[B,B^\dagger\right]+\mathrm{Tr}\left[C,C^\dagger\right]$ is the Kähler potential of the ambient space $\mathcal{S}_\Gamma$ restricted to the orbit $\mathcal{D}_\Gamma$. Indeed since $\mathcal{F}_\Gamma$ is an isometry of $\mathcal{S}_\Gamma$, the dependence in $\mathcal{K}_{\mathcal{S}}|_{\mathcal{D}}$ is only on the real part of the quiver group, namely on the real factors $\lambda_{1,2}$. Just as it stands, $\mathcal{K}_{\mathcal{S}}|_{\mathcal{D}}$ cannot work as Kähler potential of a complex Kähler metric. Yet, when the real factors $\lambda_{1,2}$ will be turned into functions of the complex coordinates z_i, then $\mathcal{K}_{\mathcal{S}}|_{\mathcal{D}}$ will be enabled to play the role of a contribution to the Kähler potential of the resolved manifold $\mathcal{M}_\zeta$.

Next we calculate the moment maps according to the formulas:

$$
\begin{aligned}
\mathfrak{P}^1 &\equiv -i\,\mathrm{Tr}\left[T^1\left(\left[A,A^\dagger\right]+\left[B,B^\dagger\right]+\left[C,C^\dagger\right]\right)\right] = e^{-2(\lambda_1+\lambda_2)}\left(1-2e^{6\lambda_1}+e^{6\lambda_2}\right)\left(\Delta_1^2+\Delta_2^2+\Delta_3^2\right) \\
\mathfrak{P}^2 &\equiv -i\,\mathrm{Tr}\left[T^2\left(\left[A,A^\dagger\right]+\left[B,B^\dagger\right]+\left[C,C^\dagger\right]\right)\right] = e^{-2(\lambda_1+\lambda_2)}\left(1+e^{6\lambda_1}-2e^{6\lambda_2}\right)\left(\Delta_1^2+\Delta_2^2+\Delta_3^2\right)
\end{aligned}
\tag{9.27}
$$

9.1.10 Solution of the moment map equations

In order to solve the moment map equations it is convenient to introduce the new variables

$$
\Upsilon_{1,2} = \exp\left[2\,\lambda_{1,2}\right] \tag{9.28}
$$

and to redefine the moment maps with indices lowered by means of the inverse of the Cartan matrix mentioned above

$$
\mathfrak{P}_i = \left(\mathfrak{C}^{-1}\right)_{\mathrm{ij}}\mathfrak{P}^j \tag{9.29}
$$

In this way imposing the level condition

$$
\mathfrak{P}_i = -\zeta_i \tag{9.30}
$$

where $\zeta_{1,2} > 0$ are the two level parameters, we obtain the final pair of algebraic equations for the factors $\Upsilon_{1,2}$

$$
\left\{\frac{\Sigma\left(-1+\Upsilon_1^3\right)}{\Upsilon_1\Upsilon_2}, \frac{\Sigma\left(-1+\Upsilon_2^3\right)}{\Upsilon_1\Upsilon_2}\right\} = \{\zeta_1,\zeta_2\} \tag{9.31}
$$

where we have introduced the shorthand notation:

$$\Sigma = \sum_{i=1}^{3} |z_i|^2 \tag{9.32}$$

The above algebraic system composed of two cubic equations is simple enough in order to find all of its nine roots by means of Cardano's formula. The very pleasant property of these solutions is that one and only one of the nine branches satisfies the correct boundary conditions, namely provides real $\Upsilon_i(\zeta,\Sigma)$ that are positive for all values of Σ and ζ and reduce to 1 when $\zeta \to 0$.

The complete solution of the algebraic equations can be written in the following way. For the first factor we have:

$$\Upsilon_1 = \frac{1}{6^{1/3}} \left(\frac{N}{\Sigma^3 \Pi^{\frac{1}{3}}}\right)^{\frac{1}{3}} \tag{9.33}$$

where

$$\begin{aligned} N &= 2\times 2^{1/3}\zeta_1^3\zeta_2^2 + 6\Sigma^3\Pi^{\frac{1}{3}} + 2\zeta_1^2\left(3\times 2^{1/3}\Sigma^3 + \zeta_2\Pi^{\frac{1}{3}}\right) + \zeta_1\left(6\times 2^{1/3}\Sigma^3\zeta_2 + 2^{2/3}\Pi^{\frac{2}{3}}\right) \\ \Pi &= 27\Sigma^6 + 9\Sigma^3\zeta_1^2\zeta_2 + 9\Sigma^3\zeta_1\zeta_2^2 + 2\zeta_1^3\zeta_2^3 + 3\sqrt{3}\Sigma^3\mathfrak{R} \\ \mathfrak{R} &= \sqrt{27\Sigma^6 + 6\Sigma^3\zeta_1\zeta_2^2 - \zeta_1^4\zeta_2^2 - 4\Sigma^3\zeta_2^3 + \zeta_1^3\left(-4\Sigma^3 + 2\zeta_2^3\right) + \zeta_1^2\left(6\Sigma^3\zeta_2 - \zeta_2^4\right)} \end{aligned} \tag{9.34}$$

For the second factor we have

$$\Upsilon_2 = \frac{-\frac{M^{8/3}}{\Sigma^5} + \frac{18M^{5/3}}{\Sigma^2} - 72M^{2/3}\Sigma + 36\left(\frac{M}{\Sigma^3}\right)^{2/3}\zeta_1^3 - 36\left(\frac{M}{\Sigma^3}\right)^{2/3}\zeta_1^2\zeta_2 + 6\left(\frac{M}{\Sigma^3}\right)^{5/3}\zeta_1^2\zeta_2}{36\times 6^{2/3}\Sigma^2\zeta_1} \tag{9.35}$$

where

$$M = \frac{6\Sigma^3\Pi^{1/3} + 2^{2/3}\Pi^{2/3}\zeta_1 + 6\times 2^{1/3}\Sigma^3\zeta_1^2 + 6\times 2^{1/3}\Sigma^3\zeta_1\zeta_2 + 2\Pi^{1/3}\zeta_1^2\zeta_2 + 2\times 2^{1/3}\zeta_1^3\zeta_2^2}{\Omega^{1/3}} \tag{9.36}$$

9.2 Discussion of cohomology in the master model

Since the two scale factors $\Upsilon_{1,2}$ are functions only of Σ, the two (1,1)-forms, relative to the two tautological bundles, respectively associated with the first and second nontrivial irreps of the cyclic group, defined in eq. (8.31) take the following general appearance:

$$\begin{aligned} \omega_{1,2}^{(1,1)} &= \frac{i}{2\pi}\left(\frac{d}{d\Sigma}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] d\bar{z}^i \wedge dz^i + \frac{d^2}{d\Sigma^2}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] z^j \bar{z}^i dz^i \wedge d\bar{z}^j\right) \\ &= \frac{i}{2\pi}\left(f_{1,2}\Theta + g_{1,2}\Psi\right) \end{aligned} \tag{9.37}$$

where we have introduced the short hand notation

$$\Theta = \sum_{i=1}^{3} d\bar{z}^i \wedge dz^i \quad ; \quad \Psi = \sum_{i,j=1}^{3} z^j \bar{z}^i dz^i \wedge d\bar{z}^j \tag{9.38}$$

Indeed in the present case the fiber metrics $\mathfrak{H}_{1,2}$ are one-dimensional and given by $\mathfrak{H}_{1,2} = \sqrt{\Upsilon_{1,2}}$. The most relevant point is that the two functions $f_{1,2}$ and $g_{1,2}$ being the derivatives (first and second) of $\Upsilon_{1,2}$ depend only on the variable Σ.

It follows that a triple wedge product of the two–forms $\omega_a^{(1,1)}$ (a=1,2) has always the following structure:

$$\omega_a^{(1,1)} \wedge \omega_b^{(1,1)} \wedge \omega_b^{(1,1)} = \left(\frac{i}{2\pi}\right)^3 \left(f_a f_b f_c + 2\Sigma\left(g_a f_b f_c + g_b f_c f_a + g_c f_a f_b\right)\right) \times \mathrm{Vol} \tag{9.39}$$

where

$$\mathrm{Vol} = dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \tag{9.40}$$

This structure enables us to calculate intersection integrals of the considered forms very easily. It suffices to change variables as we explain below. The equations

$$\Sigma = \sum_{i=1}^{3} |z_i|^2 = \rho^2 \tag{9.41}$$

define 5-spheres of radius ρ. Introducing the standard Euler angle parametrization of a 5-sphere, the volume form (9.40) reduces to:

$$\mathrm{Vol} = 8i\rho^5 \cos^4(\theta_1)\cos^3(\theta_2)\cos^2(\theta_3)\cos(\theta_4) \prod_{i=1}^{5} d\theta_i \tag{9.42}$$

The integration on the Euler angles can be easily performed and we obtain:

$$\prod_{i}^{4} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_i \int_0^{2\pi} d\theta_5 \left(8i\rho^5 \cos^4(\theta_1)\cos^3(\theta_2)\cos^2(\theta_3)\cos(\theta_4)\right) = 8i\pi^3\rho^5 \tag{9.43}$$

Hence defining the intersection integrals:

$$\mathscr{I}_{abc} = \int_{\mathscr{M}} \omega_a^{(1,1)} \wedge \omega_b^{(1,1)} \wedge \omega_b^{(1,1)} \tag{9.44}$$

we arrive at

$$\begin{aligned} \mathscr{I}_{\mathrm{abc}} &= \left(\frac{i}{2\pi}\right)^3 \times 8i\pi^3 \times \int_0^\infty \left(6\rho^5 f_a f_b f_c + 2\rho^7 \left(f_b f_c g_a + f_a f_c g_b + f_a f_b g_c\right)\right) d\rho \\ &= \int_0^\infty \left(6\rho^5 f_a f_b f_c + 2\rho^7 \left(f_b f_c g_a + f_a f_c g_b + f_a f_b g_c\right)\right) d\rho \end{aligned} \tag{9.45}$$

We have performed the numerical integration of these functions and we have found the following results

$$\begin{array}{lcl} (\zeta_1 > 0, \zeta_2 = 0) & : & \mathscr{I}_{111} = \frac{1}{8} \\ (\zeta_1 = 0, \zeta_2 \geq 0) & : & \mathscr{I}_{111} = 0 \\ (\zeta_1 > 0, \zeta_2 > 0) & : & \mathscr{I}_{111} = 1 \end{array} \tag{9.46}$$

From this we reach the following conclusion. Since the corresponding integral is nonzero it follows that:

$$\omega_S^{(2,2)} \equiv \omega_1^{(1,1)} \wedge \omega_1^{(1,1)} \tag{9.47}$$

is closed but not exact and by Poincaré duality it is the Poincaré dual of some cycle S $\in H_2(\mathscr{M})$ such that:

$$\int_S \iota^* \omega_1^{(1,1)} = \int_{\mathscr{M}} \omega_1^{(1,1)} \wedge \omega_S^{(2,2)} \tag{9.48}$$

where

$$\iota : S \longrightarrow \mathscr{M} \tag{9.49}$$

is the inclusion map. Since $H^2_c(\mathscr{M})$= $H^2(\mathscr{M})$ and both have dimension 1 it follows that dim $H_2(\mathscr{M})$ = 1, so that every nontrivial cycle S is proportional (as homology class) via some coefficient α to a single cycle $\mathscr{C}$, namely we have S= $\alpha\, \mathscr{C}$. Then we can interpret eq. (29) as follows

$$\int_{\alpha\mathscr{C}} \iota^* \omega_1^{(1,1)} = \alpha \int_{\mathscr{M}} \omega_1^{(1,1)} \wedge \omega_{\mathscr{C}}^{(2,2)} \tag{9.50}$$

If we choose as fundamental cycle, that one for which

$$\int_{\mathscr{C}} \iota^* \omega_1^{(1,1)} = 1 \tag{9.51}$$

we conclude that

$$\alpha = \left\{ \begin{array}{c|l} 1 & \text{case}\,\{\zeta_1 > 0, \zeta_2 > 0\} \\ \hline \frac{1}{8} & \text{case}\,\{\zeta_1 > 0, \zeta_2 = 0\} \end{array} \right. \tag{9.52}$$

Next we have calculated the intersection integral $\mathscr{I}_{211}$ and we have found:

$$\begin{array}{lcl} (\zeta_1 > 0, \zeta_2 = 0) & : & \mathscr{I}_{211} = 0 \\ (\zeta_1 = 0, \zeta_2 \geq 0) & : & \mathscr{I}_{211} = 0 \\ (\zeta_1 > 0, \zeta_2 > 0) & : & \mathscr{I}_{211} = 1 \end{array} \tag{9.53}$$

Conclusions on cohomology. We have two cases.

case $\{\zeta_1 > 0, \zeta_2 > 0\}$. The the first Chern classes of the two tautological bundles are cohomologous:

$$\left[\omega_1^{(1,1)}\right] = \left[\omega_2^{(1,1)}\right] = \left[\omega^{(1,1)}\right] \tag{9.54}$$

case $\{\zeta_1 > 0, \zeta_2 = 0\}$. The the first Chern class of the first tautological bundle is nontrivial and generates $H_c^{(1,1)}(\mathscr{M}) = H^{1,1}(\mathscr{M})$.

$$\left[\omega_1^{(1,1)}\right] = \text{ nontrivial} \tag{9.55}$$

The the first Chern class of the second tautological bundle is trivial , namely

$$\omega_2^{(1,1)} = \text{ exact form} \tag{9.56}$$

Obviously since there is symmetry in the exchange of the first and second scale factors, exchanging $\zeta_1 \leftrightarrow \zeta_2$, the above conclusion is reversed in the case $\{\zeta_1 = 0, \zeta_2 > 0\}$.

In passing we have also proved that the unique (2,2)-class is just the square of the unique (1,1)-class

$$\left[\omega^{(2,2)}\right] = \left[\omega^{(1,1)}\right] \wedge \left[\omega^{(1,1)}\right] \tag{9.57}$$

9.2.1 The exceptional divisor

Finally let us discuss how we retrieve the exceptional divisor $\mathbb{P}^2$ predicted by the weighted blowup argument. As we anticipated in eqs. (8.74-8.75), replacing the three coordinates z_i with

$$z_1 = W \quad ; \quad z_2 = W\,\xi_1 \quad ; \qquad z_3 = W\,\xi_2 \tag{9.58}$$

which is the appropriate change for one of the three standard open charts of $\mathbb{P}^2$, we obtain

$$\mathfrak{H}_1(\Sigma) = \frac{1}{|W|^2 H_1(\xi,\overline{\xi},W,\overline{W})} \tag{9.59}$$

where the function $H_1(\xi,\overline{\xi},W,\overline{W})$ has the property that:

$$\lim_{W\to 0} \log[H_1(\xi,\overline{\xi},W,\overline{W})] = -\log[1+|\xi_1|^2+|\xi_2|^2]+\log[\text{const}] \tag{9.60}$$

From the above result we conclude that the exceptional divisor $\mathfrak{D}^{(E)}$ is indeed the locus $W = 0$ and that on this locus the first Chern class of the first tautological bundle reduces to the Kähler 2-form of the Fubini-Study Kähler metric on $\mathbb{P}^2$. Indeed we can write:

$$c_1\,(\mathscr{L}_1)\,|_{\mathfrak{D}^{(E)}} = -\frac{i}{2\pi}\,\overline{\partial}\,\partial\,\log[1+|\xi_1|^2+|\xi_2|^2] \tag{9.61}$$

From this point of view this master example is the perfect three-dimensional analogue of the Eguchi-Hanson space, the $\mathbb{P}^1$ being substituted by a $\mathbb{P}^2$.

9.3 The model $\frac{\mathbb{C}^3}{\mathbb{Z}_4}$

Another interesting model is the case $\frac{\mathbb{C}^3}{\mathbb{Z}_4}$, where the group $\mathbb{Z}_4$ is generated by

$$Y = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{9.62}$$

9.3.1 Construction of the blowups according to conjugacy classes

Calculating the ages we find two junior conjugacy classes, namely $\frac{1}{4}$ $\{1,\ 1,\ 2\}$ and $\frac{1}{4}$ $\{2{,}2{,}0\}$, and one senior class $\frac{1}{4}\{3{,}3{,}2\}$. Hence we expect two cohomology classes of type (1,1) and one cohomology class of type (2,2). The fact that there is a senior cohomology class means that one of the two exceptional divisors is compact, and the other is not. One of the consequences of all this is that out of the three tautological line bundles that we construct solving the moment map equation one must be dependent on the other two.
According to the theory of weighted blowup we introduce the following two sets of coordinates corresponding to the two junior classes

$$\Psi_{[1,1,2]} = W\left(1,\Psi_1,\sqrt{\Psi_3}\right);\Phi_{[2,2,0]} = W^2\,(1,\Phi_1)\times\Phi_2 \tag{9.63}$$

In the first case Ψ_1 and Ψ_3 are the inhomogenous coordinates on $\mathbb{WP}^{1,1,2}$, while in the second case Φ_1 is the inhomogeneous coordinate on $\mathbb{P}^1$, while Φ_2 spans $\mathbb{C}$.

The three moment maps $\mathfrak{P}_i$ can be calculated exactly with the same procedure as in the previous master case and as in the cyclic examples of appendices D. It is also convenient to rearrange the moment maps as follows

$$\Pi_1 = \mathfrak{P}_1 - \mathfrak{P}_3 \; ; \; \Pi_2 = \mathfrak{P}_1 + \mathfrak{P}_3 - \mathfrak{P}_2 \; ; \; \Pi_3 = -\mathfrak{P}_2 \tag{9.64}$$

This is done by the following nonsingular matrix

$$S = \left(\begin{array}{c|c|c} 1 & 0 & -1 \\ \hline 1 & -1 & 1 \\ \hline 0 & 1 & 0 \end{array}\right) \; ; \; \Pi = S.\mathfrak{P} \tag{9.65}$$

Furthermore it is convenient to introduce reduced variables $\mathfrak{H}_i$=$\sqrt{X_i}$ and new level parameters κ_i = $S_{\mathrm{ij}}\zeta_j$

$$\{\kappa_1, \kappa_2, \kappa_3\} = \{\zeta_1 - \zeta_3, \zeta_1 - \zeta_2 + \zeta_3, \zeta_2\} \tag{9.66}$$

In terms of these variables the moment map equation hence takes the following form:

$$\left(\begin{array}{c} -\frac{\left(X_1^2-X_3^2\right)\left(X_1X_3\left(\Delta_1^2+\Delta_2^2\right)+\left(1+X_2^2\right)\Delta_3^2\right)}{X_1X_2X_3} \\ \frac{\left(X_2+X_2^3-X_1X_3\left(X_1^2+X_3^2\right)\right)\left(\Delta_1^2+\Delta_2^2\right)}{X_1X_2X_3} \\ -\frac{\left(-1+X_2^2\right)\left(X_2\left(\Delta_1^2+\Delta_2^2\right)+\left(X_1^2+X_3^2\right)\Delta_3^2\right)}{X_1X_2X_3} \end{array}\right) = \left(\begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{array}\right) \tag{9.67}$$

where $\Delta_i = \mid z_i \mid$ are the moduli of the three complex coordinates.

9.3.2 Proof that one tautological line bundle depends from the other two

By this we mean that the isomorphism classes of these bundles are not linearly independent in the Picard group. We shall indeed show that the differential forms representing the first Chern classes of the three bundles satisfy a linear relation. The strategy we adopt to check this fact is the following. We observe that the equations are written in such a way that they depend only on two variables Z= $|z_1|^2$+$|z_2|^2$ and U=$|z_3|^2$. Hence instead of solving the equations for $X_{1,2,3}$, in terms of the levels $\kappa_{1,2,3}$and of U and Z, we rather do the reverse and we solve them for U and Z in terms of the levels $\kappa_{1,2,3}$ and of $X_{1,2,3}$.. Just because there are three independent equations for two variables, by substituting back we obtain a condition that has to be satisfied by $\mathfrak{H}_{1,2,3} = \sqrt{X_{1,2,3}}$ and $\kappa_{1,2,3}$, among themselves which is the following one:

$$\mathfrak{H}_3 = \mathfrak{H}_1 \sqrt[4]{\frac{\kappa_1 - \kappa_2 + \kappa_3 + (-\kappa_1 + \kappa_2 + \kappa_3)\,\mathfrak{H}_2^4}{-\kappa_1 - \kappa_2 + \kappa_3 + (\kappa_1 + \kappa_2 + \kappa_3)\,\mathfrak{H}_2^4}} \tag{9.68}$$

which implies

$$\mathrm{Log}\left[\mathfrak{H}_3\right] = \mathrm{Log}\left[\mathfrak{H}_1\right] + \frac{1}{4}\mathrm{Log}\left[-a + \mathfrak{H}_2^4\right] - \frac{1}{4}\mathrm{Log}\left[-b + \mathfrak{H}_2^4\right] \tag{9.69}$$

where a, b are two constants

$$a = -\frac{\kappa_1 - \kappa_2 + \kappa_3}{-\kappa_1 + \kappa_2 + \kappa_3} \quad ; \quad b = -\frac{-\kappa_1 - \kappa_2 + \kappa_3}{\kappa_1 + \kappa_2 + \kappa_3} \tag{9.70}$$

Expanding in power series of a small parameter we have:

$$\begin{aligned}
&\kappa_1 = \varepsilon k_1 + \mathscr{O}[\varepsilon^2] \quad , \quad \kappa_2 = \varepsilon k_2 + \mathscr{O}[\varepsilon^2] \quad , \quad \kappa_3 = \varepsilon k_3 + \mathscr{O}[\varepsilon^2] \\
&\mathfrak{H}_1 = 1 - \varepsilon \omega_1 + \mathscr{O}[\varepsilon]^2 \\
&\mathfrak{H}_2 = 1 - \varepsilon \omega_2 + O[\varepsilon]^2 \\
&\mathfrak{H}_3 = 1 - \varepsilon \omega_3 + \mathscr{O}[\varepsilon^2]
\end{aligned} \tag{9.71}$$

we calculate to that order the first Chern classes of the line bundles and we find:

$$\omega_1^{(1,1)} = \frac{i}{2\pi} \partial\overline{\partial}\mathrm{Log}\left[1 - \varepsilon\, \omega_1 + \mathscr{O}[\varepsilon^2]\right]$$

$$\omega_2^{(1,1)} = \frac{i}{2\pi} \partial\overline{\partial}\mathrm{Log}\left[1 - \varepsilon\, \omega_2 + \mathscr{O}[\varepsilon^2]\right] \tag{9.72}$$

$$\omega_3^{(1,1)} = \frac{i}{2\pi} \partial\overline{\partial}\mathrm{Log}\left[1 - \varepsilon\, \omega_3 + \mathscr{O}[\varepsilon^2]\right] \tag{9.73}$$

where

$$\begin{aligned}
\omega_1 &= \frac{i}{2\pi} \frac{2k_1\left(|z_1|^2+|z_2|^2\right)+2k_3\left(|z_1|^2+|z_2|^2\right)+k_2\left(|z_1|^2+|z_2|^2+2|z_3|^2\right)}{\left(16\left(|z_1|^2+|z_2|^2\right)\left(|z_1|^2+|z_2|^2+2|z_3|^2\right)\right)} \\
\omega_2 &= \frac{i}{2\pi} \frac{\varepsilon k_3}{4\left(|z_1|^2+|z_2|^2+2|z_3|^2\right)} \\
\omega_3 &= \frac{i}{2\pi} \frac{\left(-2k_1\left(|z_1|^2+|z_2|^2\right)+2k_3\left(|z_1|^2+|z_2|^2\right)+k_2\left(|z_1|^2+|z_2|^2+2|z_3|^2\right)\right)}{16\left(|z_1|^2+|z_2|^2\right)\left(|z_1|^2+|z_2|^2+2|z_3|^2\right)}
\end{aligned} \tag{9.74}$$

This solution perfectly agrees with the prediction on the relation between Chern classes at this order that follows from the relation (9.68) between $\mathfrak{H}$ factors:

$$\mathrm{Log}\left[\mathfrak{H}_3\right] = \frac{1}{4}\mathrm{Log}\left[1 + \frac{2k_1}{-k_1+k_2+k_3}\right] + \left(\omega_1 - \frac{k_1\omega_2}{k_3}\right)\varepsilon + O[\varepsilon^2] \tag{9.75}$$

which implies

$$\omega_3 = \left(\omega_1 - \frac{k_1\omega_2}{k_3}\right) \tag{9.76}$$

9.3.3 Special solution of the moment map equations

If we choose the following special values of the level parameters

$$\kappa_3 = -\kappa, \kappa_2 = 0, \kappa_1 = \kappa \tag{9.77}$$

we obtain a solution of the moment map equations by setting:

$$\mathfrak{H}_3 = \mathfrak{H}_2, \mathfrak{H}_1 = 1 \tag{9.78}$$

where the $\mathfrak{H}_2$ satisfies the following quartic equation:

$$U + Z\mathfrak{H}_2 + \kappa\mathfrak{H}_2^2 - Z\mathfrak{H}_2^3 - U\mathfrak{H}_2^4 = 0 \tag{9.79}$$

Indeed with the choices (9.77) and (9.78) the moment map equations reduce to the above quartic algebraic constraint.

Thanks to Cardano's formula we have four roots only one of which has the correct property that it reduces to 1 when the level parameter κ goes to zero. Such a solution has the following explicit appearance:

$$\mathfrak{H}_2 = \frac{1}{2\sqrt{3}}\sqrt{\frac{\sqrt{6}\sqrt{A}+\sqrt{3}\sqrt{B}-3Z}{U}} \tag{9.80}$$

$$\begin{aligned} A &= 8\kappa U + 2^{2/3}\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}U + \frac{2\sqrt[3]{2}\Lambda_1 U}{\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}} \\ &\quad - \frac{3\sqrt{3}\Lambda_3}{\sqrt{8\kappa U - 2\,2^{2/3}\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}U - \frac{4\sqrt[3]{2}\Lambda_1 U}{\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}} + 3Z^2}} + 3Z^2 \end{aligned} \tag{9.81}$$

$$B = 8\kappa U - 2\,2^{2/3}\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}U - \frac{4\sqrt[3]{2}\Lambda_1 U}{\sqrt[3]{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1^3}}} + 3Z^2 \tag{9.82}$$

where $\Lambda_{1,2,3}$ is a short hand for three polynomials in U and Z that are specified below:

$$\Lambda_1 = -12U^2+3Z^2+\kappa^2, \Lambda_2 = 72U^2\kappa+9Z^2\kappa+2\kappa^3, \Lambda_3 = Z\left(-8U^2+Z^2+4U\kappa\right) \tag{9.83}$$

On the other hand Z and U are short hand notations for:

$$Z = |z_1|^2+|z_2|^2 \;\; ; \, U = |z_3|^2 \tag{9.84}$$

In this way the calculation of the first Chern classes from the irrep side is explicit in this case:

$$\omega_1^{(1,1)} = 0 \tag{9.85}$$

$$\omega_3^{(1,1)} = \omega_2^{(1,1)} = \frac{i}{2\pi}\overline{\partial}\partial \mathrm{Log}\left[\mathfrak{H}_2\right] \tag{9.86}$$

10 Conclusions

As we emphasized in the introduction the present paper focuses on the resolution of $\mathbb{C}^3/\Gamma$ singularities, $\Gamma \subset \mathrm{SU}(3)$ being a finite group, in the perspective of applications to the duality between superconformal Chern-Simons gauge theories in three space–time dimensions and M2-brane solutions of $D = 11$ supergravity. In many regards this perspective is the M-theory analogue of what was done about 18 years ago in [39,40] where the smooth analogue of fractional three-branes was considered, replacing the 6-dimensional singular transverse dimensions $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_k$ to a fractional three-brane with their smooth resolution $\mathbb{C}\times ALE_k$. Here the analogous scenario is:

$$\mathbb{C}\times\frac{\mathbb{C}^3}{\Gamma} \longrightarrow \mathbb{C}\times\mathscr{M}_\zeta \tag{10.1}$$

The main point of this paper is that the generalization to the case of three-folds of the Kronheimer construction of ALE manifolds encodes precisely, in a well-defined geometrical language, all the necessary information and the needed steps that lead to a dual superconformal Chern-Simons gauge theory on the M2-brane world-volume. A comment is in order to appreciate the difference between our results and those obtained in the ABJM set up [29]. The orbifold of the seven sphere considered in [29] corresponds to an identical phase action of the cyclic group $\mathbb{Z}_k$ on all the four homogeneous coordinates of $\mathbb{P}^3$ base of the Hopf-fibration of $\mathbb{S}^7$. The resulting quotient $\mathbb{C}^4/\mathbb{Z}_k$ pertaining to the ABJM case is not of the type considered in this paper, since that representation of $\Gamma = \mathbb{Z}_k$ is not in $\mathrm{SU}(4)$ (unless $k = 4$), but only in $\mathrm{U}(4)$. The resolution of such a singularity does not have the properties predicted by the Ito–Reid theorem, in particular, it is not crepant. The quantization of the levels of the Chern-Simons terms in terms of k, which is the characterizing features of the ABJM-case is to be rediscussed from scratch in the case of the theories addressed in the present paper. At the present stage we do not know what the analogue feature might be in our case.

A part from the above mentioned unsolved problem, we already listed the impressive translation vocabulary between geometry and field theory in the introduction and we do not repeat it once again. In this final section we rather summarize the achieved results, the open problems and the perspectives for further investigations.

A) One important issue concerns the difference between hyperKähler and Kähler quotients, which is also the difference between $\mathcal{N} = 4/\mathcal{N} = 3$ supersymmetry and $\mathcal{N} = 2$ supersymmetry in three space–time dimensions. As we extensively discussed in the main text, there is an analogue of the holomorphic part of moment map equation that is provided by the equation $\boldsymbol{p}\wedge\boldsymbol{p} = 0$, defining the subspace $\mathscr{D}_\Gamma \subset \mathrm{Hom}_\Gamma(Q\times R, R)$ (see eqs. (6.10-6.12)) and leads to a precise suggestion for the superpotential $\mathscr{W}$ in the corresponding superconformal field theory. A striking feature of this generalization that cannot escape notice is the following. In the hyperKähler case the holomorphic moment map equation is of the form $[p_1, p_2] = 0$ (eventually deformed by the ζ^+ parameters). In the Kähler case it is $\varepsilon^{xyz} p_x \cdot p_y = 0$ with $x, y, z = 1, 2, 3$. It looks like we have a *reversed correspondence* with division algebras, the hyperKähler case corresponding to $\mathbb{C}$ (one imaginary unit, hence one matrix condition), the Kähler case corresponding instead to quaternions (three imaginary units, hence three matrix conditions). This naturally leads to a wild conjecture. What about a further generalization of the Kronheimer construction corresponding to octonions? We might consider finite groups $\Gamma \subset \mathrm{SU}(4)$ that are also subgroups of $\mathrm{G}_{2(-14)}$, namely preserve the octonion structure constants a^{ijk}, and we might consider 7-tuples of matrices p_i $(i = 1,\dots,7)$ each belonging to $\mathrm{Hom}(\mathrm{R},\mathrm{R})$, where R denotes the regular representation of Γ. Because of our hypothesis there is a natural 7-dimensional representation $\mathscr{Q}$ of Γ (corresponding to the embedding $\Gamma \hookrightarrow \mathrm{G}_{2(-14)}$) and we can define the space $\widehat{\mathscr{S}_\Gamma} = \mathrm{Hom}_\Gamma(\mathscr{Q}\otimes R, R)$. Next one can deem of the analogue subspace $\widehat{D}_\Gamma \subset \widehat{S}_\Gamma$ singled out by the 7 conditions:

$$a^{ijk}\, p_j \cdot p_k = 0 \tag{10.2}$$

We might perform some quotient of this space with respect to a suitable compact quiver group $\mathscr{F}_\Gamma$. What the result might be of such a construction is totally unexplored both from the mathematical and from the physical point of view. Yet it is particularly inspiring that the mentioned conditions for the finite group Γ are satisfied by L_{168} and by all of its subgroups. Careful consideration of this possibility is certainly an interesting line for further research.

B) As we repeatedly stressed, the relation between the first Chern classes of the tautological bundles associated, by means of the generalized Kronheimer construction, with the nontrivial irreps of the discrete group Γ and the components of the exceptional divisor produced by the blowup is of central relevance on both sides of the correspondence Geometry/Superconformal Gauge-Theory. In the present paper this relation has been explored and fully established in one of the few cases where the algebraic moment map equations admit an explicit analytic solution, *i.e* for the master case treated in section 9.1, where there is just one junior class and the blowup produces just one component of the exceptional divisor which is a $\mathbb{P}^2$. The extension of this in depth analysis to cases where the junior classes are several and moreover to cases where Γ is nonabelian is one principal direction for further investigation and we plan to address such a question in the nearest future. This issue was fully solved in the abelian case in the paper [52] using different techniques and relying on results presented in [53].

C) In relation with the above issue a very important point concerns the possibility of analyzing algebraic moment map equations by means of a series development in the neighborhood of zero for the level parameters ζ_I. Since intersection integrals in cohomology do not depend on the value of ζ_I, except for a jump when $\zeta_I = 0$ for some value of I, we can advocate the use of infinitesimal ζ_I. A rigorous investigation of this possibility is of utmost relevance since the solution of moment map equations at first order in the ζ_I is always accessible. This might prove the winning weapon to discriminate among all the available cohomology patterns. We plan to address this question in the nearest future.

D) In the long run the main target is to promote the superconformal Chern-Simons gauge theory on the brane from the theory of one M2-brane to the theory of several M2-branes which means to add color indices to the scalar fields. These latter are the coordinates of $\mathscr{S}_\Gamma \equiv \mathrm{Hom}_\Gamma(\mathscr{Q} \otimes R, R)$ and in the case where Γ is a cyclic $\mathbb{Z}_k$, the gauge group $\mathscr{F}_\Gamma$ produced by the Kronheimer construction is just a product of $\mathrm{U}(1)$.s. Adding color indices typically amounts to promote the $\mathrm{U}(1)$'s to U(N), where N is the number of colors. It is a completely open question to establish the enlargement by colors of those nonabelian gauge groups $\mathscr{F}_\Gamma = \prod_{\mu=1}^{r} \mathrm{U}(\mathrm{n}_\mu)$ that are produced by a nonabelian discrete group Γ.

E) Last but not least in the list of open problems is a detailed analysis of the relation between the algebraic blowup procedure of singular orbifolds $\mathbb{C}^3/\Gamma$, as that utilized by Markushevich for the case $\Gamma = \mathrm{L}_{168}$ [32], and the Kähler quotient algorithm discussed in the present paper. We plan to come back to this point in the nearest future.

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A Age grading for various groups

Since our main interest is to chart the possible M2-brane solutions and their dual Chern-Simons gauge theories that can be obtained by various choices of the discrete group Γ, and since several supergravity arguments point to the fundamental relevance of the group L_{168} (see [26, 33]), in this appendix we just provide the calculation of ages for the maximal subgroups of L_{168} and their further subgroups starting from its embedding in SU(3) considered by Markushevich [32].

A.1 Ages for $\Gamma \subset \mathrm{L}_{168}$

Starting from the construction of the irreducible three-dimensional complex representation discussed in [26, 33], we have computed the ages of the various conjugacy classes for the holomorphic action of the group L_{168} on $\mathbb{C}^3$.

In order to be able to compare with Markusevich's paper [32], it is important to note that the form given by Markusevich of the generators which he calls τ , χ and ω, respectively of order 7, 3 and 2, does not correspond to the standard generators in the presentation of the group L_{168} utilized by one of us in the recent paper [26]. Yet there is no problem since we have a translation vocabulary. Setting:

$$R = \omega.\chi \quad ; \quad S = \chi.\tau \quad ; \quad T = \chi^2.\omega \tag{A.1}$$

these new generators satisfy the standard relations of the presentation displayed below:

$$\mathrm{L}_{168} = \left(R, S, T \parallel R^2 = S^3 = T^7 = RST = (TSR)^4 = \mathbf{e}\right) \tag{A.2}$$

From now on we utilize the abstract notation in terms of $\rho = R, \sigma = S, \tau = T$.

We begin by constructing explicitly the group L_{168} in Markushevich's basis substituting the analytic form of the generators which follows from the identification (A.1). We find

$$\begin{aligned}
\varepsilon &\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\rho &\to \begin{pmatrix} -\frac{2\mathrm{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} & -\frac{2\mathrm{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} & \frac{2\mathrm{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} \\ -\frac{2\mathrm{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} & \frac{2\mathrm{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} & -\frac{2\mathrm{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} \\ \frac{2\mathrm{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} & -\frac{2\mathrm{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} & -\frac{2\mathrm{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} \end{pmatrix} \\
\sigma &\to \begin{pmatrix} 0 & 0 & -(-1)^{1/7} \\ (-1)^{2/7} & 0 & 0 \\ 0 & (-1)^{4/7} & 0 \end{pmatrix} \\
\tau &\to \begin{pmatrix} \frac{i+(-1)^{13/14}}{\sqrt{7}} & -\frac{(-1)^{1/14}\left(-1+(-1)^{2/7}\right)}{\sqrt{7}} & \frac{(-1)^{9/14}\left(1+(-1)^{1/7}\right)}{\sqrt{7}} \\ \frac{(-1)^{11/14}\left(-1+(-1)^{2/7}\right)}{\sqrt{7}} & \frac{i+(-1)^{5/14}}{\sqrt{7}} & \frac{(-1)^{3/14}\left(1+(-1)^{3/7}\right)}{\sqrt{7}} \\ -\frac{(-1)^{11/14}\left(1+(-1)^{1/7}\right)}{\sqrt{7}} & -\frac{(-1)^{9/14}\left(1+(-1)^{3/7}\right)}{\sqrt{7}} & -\frac{-i+(-1)^{3/14}}{\sqrt{7}} \end{pmatrix}
\end{aligned} \tag{A.3}$$

We remind the reader that ρ, σ, τ are the abstract names for the generators of L_{168} whose 168 elements are written as words in these letters (modulo relations). Substituting these letters with explicit matrices that satisfy the defining relation of the group one obtains an explicit representation of the latter. In the present case the substitution A.3 produces the irreducible 3-dimensional representation DA_3.

A.1.1 The case of the full group $\Gamma = L_{168}$

Utilizing this explicit representation it is straightforward to calculate the age of each conjugacy class and we obtain the result displayed in the following table.

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Conjugacy class of } L_{168} & \mathscr{C}_1 & \mathscr{C}_2 & \mathscr{C}_3 & \mathscr{C}_4 & \mathscr{C}_5 & \mathscr{C}_6 \\
\hline
\text{representative of the class} & \mathbf{e} & R & S & TSR & T & SR \\
\hline
\text{order of the elements in the class} & 1 & 2 & 3 & 4 & 7 & 7 \\
\hline
\text{age} & 0 & 1 & 1 & 1 & 1 & 2 \\
\hline
\text{number of elements in the class} & 1 & 21 & 56 & 42 & 24 & 24 \\
\hline
\end{array} \tag{A.4}$$

A.1.2 The case of the maximal subgroup $\Gamma = G_{21} \subset L_{168}$

In order to obtain the ages for the conjugacy classes of the maximal subgroup G_{21}, we just need to obtain the explicit three-dimensional form of its generators $\mathscr{X}$ and $\mathscr{Y}$ satisfying the defining relations:

$$\mathscr{X}^3 = \mathscr{Y}^7 = \mathbf{1} \quad ; \quad \mathscr{X}\mathscr{Y} = \mathscr{Y}^2\mathscr{X} \tag{A.5}$$

This latter is determined by the above explicit form of the L_{168} generators, by recalling the embedding relations:

$$\mathscr{Y} = \rho\,\sigma\,\tau^3\,\sigma\,\rho \quad ; \quad \mathscr{X} = \sigma\,\rho\,\sigma\,\rho\,\tau^2 \tag{A.6}$$

In this way we obtain the following explicit result:

$$\begin{aligned}
\mathscr{Y} \to \mathrm{Y} &= \begin{pmatrix} -(-1)^{3/7} & 0 & 0 \\ 0 & (-1)^{6/7} & 0 \\ 0 & 0 & -(-1)^{5/7} \end{pmatrix} \\
\mathscr{Y} \to \mathrm{X} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{A.7}$$

Hence, for the action on $\mathbb{C}^3$ of the maximal subgroup $G_{21} \subset L_{168}$ we obtain the following ages of its conjugacy classes:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Conjugacy Class of } G_{21} & C_1 & C_2 & C_3 & C_4 & C_5 \\
\hline
\text{representative of the class} & e & \mathscr{Y} & \mathscr{X}^2\mathscr{Y}\mathscr{X}\mathscr{Y}^2 & \mathscr{Y}\mathscr{X}^2 & \mathscr{X} \\
\hline
\text{order of the elements in the class} & 1 & 7 & 7 & 3 & 3 \\
\hline
\text{age} & 0 & 2 & 1 & 1 & 1 \\
\hline
\text{number of elements in the class} & 1 & 3 & 3 & 7 & 7 \\
\hline
\end{array} \tag{A.8}$$

A.1.3 The case of the two maximal octahedral subgroups

For the other two maximal subgroups $\mathrm{O}_{24\mathrm{A}}$ and $\mathrm{O}_{24\mathrm{B}}$ we find instead an identical result. This is retrieved from the two embedding conditions of the generators S and T, satisfying the defining relations:

$$S^2 = T^3 = (ST)^4 = \mathbf{1} \tag{A.9}$$

Subgroup $\mathrm{O}_{24\mathrm{A}}$

$$T = \rho\,\sigma\,\rho\,\tau^2\,\sigma\,\rho\,\tau \quad ; \quad S = \tau^2\,\sigma\,\rho\,\tau\,\sigma^2 \tag{A.10}$$

Subgroup $\mathrm{O}_{24\mathrm{B}}$

$$T = \rho\,\tau\,\sigma\,\rho\,\tau^2\,\sigma\,\rho\,\tau \quad ; \quad S = \sigma\,\rho\,\tau\,\sigma\,\rho\,\tau \tag{A.11}$$

In this way we get:

Conjugacy Class of the $\mathrm{O}_{24\mathrm{A}}$	C_1	C_2	C_3	C_4	C_5
representative of the class	e	T	$STST$	S	ST
order of the elements in the class	1	3	2	2	4
age	0	1	1	1	1
number of elements in the class	1	8	3	6	6

(A.12)

and

Conjugacy Class of the $\mathrm{O}_{24\mathrm{B}}$	C_1	C_2	C_3	C_4	C_5
representative of the class	e	T	$STST$	S	ST
order of the elements in the class	1	3	2	2	4
age	0	1	1	1	1
number of elements in the class	1	8	3	6	6

(A.13)

A.1.4 The case of the cyclic subgroups $\mathbb{Z}_3$ and $\mathbb{Z}_7$

Last we consider the age grading for the quotient singularities $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_7$. As generators of the two cyclic groups we respectively choose the matrices X and Y displayed in eq. (A.7). In other words we utilize either one of the two generators of the maximal subgroup $\mathrm{G}_{21} \subset \mathrm{L}_{168}$.

The $\Gamma = \mathbb{Z}_3$ case. The first step consists of diagonalizing the action of the generator X. Introducing the unitary matrix:

$$\mathfrak{q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1+i\sqrt{3}}{2\sqrt{3}} & \frac{-1-i\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1-i\sqrt{3}}{2\sqrt{3}} & \frac{-1+i\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{A.14}$$

we obtain:

$$\widetilde{\mathrm{X}} \equiv \mathfrak{q}^\dagger\,\mathrm{X}\,\mathfrak{q} = \begin{pmatrix} e^{\frac{2i\pi}{3}} & 0 & 0 \\ 0 & e^{-\frac{2i\pi}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.15}$$

This shows that the quotient singularity $\mathbb{C}^3/\mathbb{Z}_3$ is actually of the form $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ since it suffices to change basis of $\mathbb{C}^3$ by introducing the new complex coordinates:

$$\widetilde{z}_a = \mathfrak{q}_a{}^b z_b \tag{A.16}$$

It follows that in the resolution of the singularity we will obtain:

$$ALE_{\mathbb{Z}_3} \times \mathbb{C} \to \frac{\mathbb{C}^3}{\mathbb{Z}_3} \tag{A.17}$$

Yet, as we discuss more extensively below, the starting setup $\mathbb{C}^3/\Gamma$ produces a special type of ALE-manifold where all the holomorphic moment map levels are frozen to zero and only the Kähler quotient parameters are switched on.

Eq. (A.15) corresponds also to the decomposition of the three-dimensional representation of Z_3 into irreducible representations of $\mathbb{Z}_3$. From the diagonalized form (A.15) of the generator we immediately obtain the ages of the conjugacy classes:

Conjugacy Class of $\mathbb{Z}_3$	C_1	C_2	C_3
representative of the class	e	X	X^2
order of the elements in the class	1	3	3
age	0	1	1
number of elements in the class	1	1	1

(A.18)

The $\Gamma = \mathbb{Z}_7$ case. In the $\mathbb{Z}_7$ case, the generator Y is already diagonal and, as we see, none of the three complex coordinates is invariant under the action of the group. Hence differently from the previous case we obtain:

$$\mathscr{M}_{\mathbb{Z}_7} \to \frac{\mathbb{C}^3}{\mathbb{Z}_7} \tag{A.19}$$

where the resolved smooth manifold is not the direct product of $\mathbb{C}$ with an ALE-manifold:

$$\mathscr{M}_{\mathbb{Z}_7} \neq ALE_{\mathbb{Z}_7} \times \mathbb{C} \tag{A.20}$$

From the explicit diagonal form (A.7) of the generator we immediately obtain the ages of the conjugacy classes:

Conjugacy Class of $\mathbb{Z}_7$	C_1	C_2	C_3	C_4	C_5	C_6	C_7
representative of the class	e	Y	Y^2	Y^3	Y^4	Y^5	Y^6
order of the elements in the class	1	7	7	7	7	7	7
age	0	2	2	1	2	1	1
number of elements in the class	1	1	1	1	1	1	1

(A.21)

B The McKay quiver of various groups

Since it is of central relevance to the resolution of the singularity by means of a Kähler quotient based on a generalized Kronheimer construction, it is convenient to calculate also the Mckay quiver matrices associated to the subgroups considered in the previous appendix.

B.1 The McKay quiver of L_{168}

We calculate the McKay matrix defined by

$$\mathscr{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{6} \mathscr{A}_{ij} \mathrm{D}_j \tag{B.1}$$

where $\mathscr{Q}$ is the three-dimensional complex representation defining the action of L_{168} on $\mathbb{C}^3$ while D_i denote the 6 irreducible representation ordered in the standard way we have so far adopted, namely:

$$\mathrm{D}_i = \left\{ \mathrm{D}_1, \mathrm{D}_6, \mathrm{D}_7, \mathrm{D}_8, \mathrm{D}_3, \mathrm{D}_{\overline{3}} \right\} \tag{B.2}$$

We find the following matrix:

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{B.3}$$

The matrix $\mathscr{A}$ admits the graphical representation displayed in fig.6, named the *McKay quiver* of the quotient $\mathbb{C}^3/\mathrm{L}_{168}$ [8]

B.2 The McKay quiver of G_{21}

We calculate the McKay matrix defined by

$$\mathscr{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{5} \mathscr{A}_{ij} \mathrm{D}_j \tag{B.4}$$

where $\mathscr{Q}$ is the three-dimensional complex representation defining the action of G_{21} on $\mathbb{C}^3$ while D_i denote the 5 irreducible representations ordered in the standard way we have so far adopted, namely:

$$\mathrm{D}_i = \left\{ \mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_{\overline{1}}, \mathrm{D}_3, \mathrm{D}_{\overline{3}} \right\} \tag{B.5}$$

[8] The authors are grateful to their friend Massimo Bianchi who noticed that the McKay quiver presented in the ArXiv version of the present paper was probably mistaken since the number of lines entering and outgoing from some of the nodes was not equal. This caused a revision of our calculation of the quiver matrix (B.3) where there was indeed a trivial, yet hidden, mistake. The mistake was spotted by Bianchi because of his experience with quiver gauge theories discussed in the several papers [54–56]. What is not widely appreciated in this literature and in general in the physical community is that the quivers utilized in such a gauge–theory capacity and the McKay quivers, group theoretically defined as we do in the present paper, are just one and the same thing. This point will be addressed in full generality in a forthcoming paper [57].

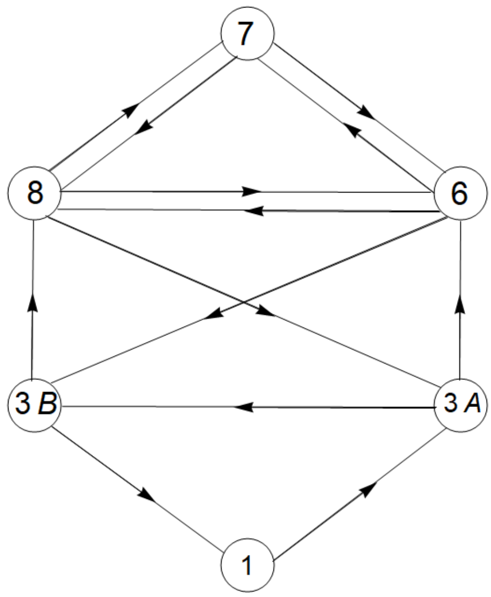

Figure 6: The quiver diagram of the finite group $L_{168} \subset SU(3)$

We find the following matrix:

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{B.6}$$

The matrix $\mathscr{A}$ admits the graphical representation presented in fig. 7, named the *McKay quiver* of the quotient $\mathbb{C}^3/G_{21}$.

B.3 The McKay quiver for $\mathbb{Z}_3$

Next we calculate the McKay matrix for the case where $\mathscr{Q}$ is the three-dimensional representation of the group Z_3 generated by X given as in eq. (A.7) and D_i are the three irreducible one-dimensional representations of $\mathbb{Z}_3$. The result is displayed below.

$$\mathscr{Q} \otimes D_i = \bigoplus_{j=1}^{3} \mathscr{A}_{ij} D_j \tag{B.7}$$

$$\mathscr{A}_{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{B.8}$$

The matrix A_{ij} in eq. (B.8) admits the graphical representation shown in eq. (B.8).

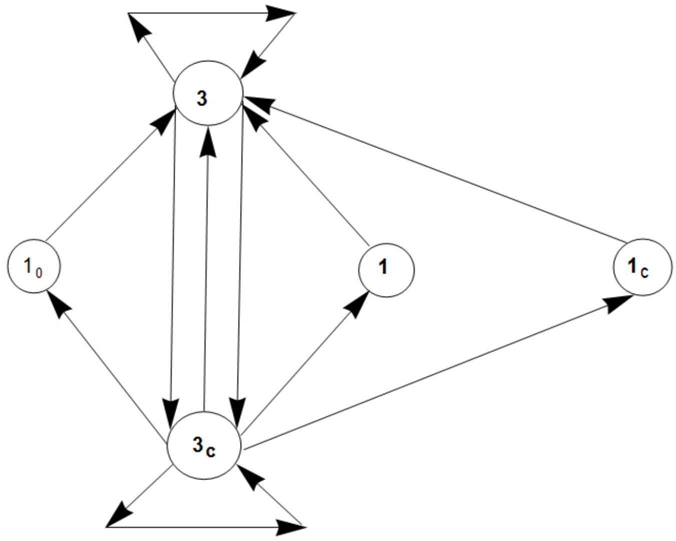

Figure 7: The quiver diagram of the finite group $G_{21} \subset L_{168}$

B.4 The McKay quiver for $\mathbb{Z}_7$

Then we calculate the McKay matrix for the case where $\mathscr{Q}$ is the three-dimensional representation of the group $\mathbb{Z}_7$ generated by Y as given in eq. (A.7) and D_i are the seven irreducible one-dimensional representations of $\mathbb{Z}_7$. The result is displayed below.

$$\mathscr{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{7} \mathscr{A}_{ij} \mathrm{D}_j \tag{B.9}$$

$$\mathscr{A}_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{B.10}$$

The McKay matrix in eq. (B.10) admits the graphical representation of fig.9.

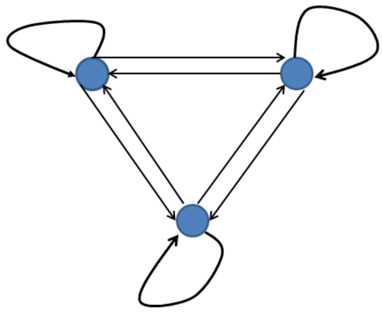

Figure 8: The quiver diagram of the finite group $\mathbb{Z}_3 \subset \mathrm{L}_{168}$. The three vertices correspond to the three irreducible representations, $\mathbf{1}$, ψ and ψ^2, where ψ is a primitive cubic root of unity. It is not necessary to mark the names of the representation since the quiver diagram is completely symmetric. In each vertex converge three lines and three lines depart from each vertex. It is interesting to compare this quiver diagram with that of the master model presented in fig.5.

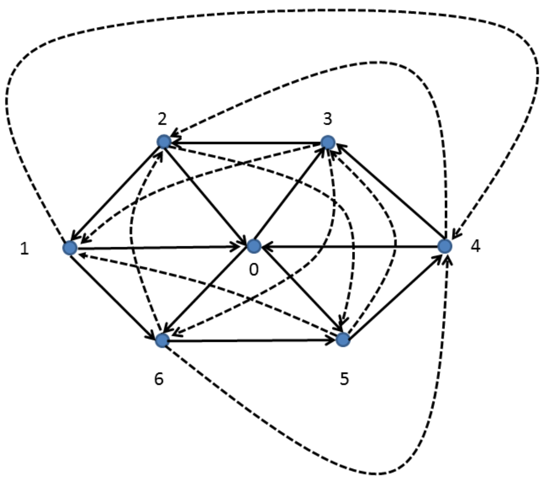

Figure 9: The quiver diagram of the finite group $\mathbb{Z}_7 \subset \mathrm{L}_{168}$. The seven vertices correspond to the seven irreducible representations, $\mathbf{1}$, marked 0, and $\psi, \ldots, \psi^6$, marked $1, 2, \ldots, 6$, where ψ is a primitive seventh root of the unity. In each vertex converge three lines and three lines depart from each vertex.

C Gibbons-Hawking metrics and the resolution of $\mathbb{C}^2/\Gamma$ singularities

As an illustration of the above outlined generalized Kronheimer construction which resolves the quotient singularities $\mathbb{C}^3/\Gamma$, we intend to discuss the abelian cases $\Gamma = \mathbb{Z}_3$ and $\mathbb{Z}_7$, looking at the case where such finite groups are subgroups of $\mathrm{L}_{168} \subset \mathrm{SU}(3)$. In this way, by steps of increasing complexity, we approach the discussion of the nonabelian cases like G_{21}. When $\Gamma = \mathbb{Z}_3 \subset \mathrm{L}_{168} \subset \mathrm{SU}(3)$ we already pointed out that the singularity is actually of the type mentioned in eq. (A.17). This is quite useful for our purposes since the $ALE_{\mathbb{Z}_k}$ manifolds admit another well known representation with which we can compare the Kronheimer construction in order to get orientation in our main task of understanding the cohomology of the resolved Kähler manifold. The representation we are alluding to is that of the Gibbons-Hawking multicenter metrics that are known to be hyperKählerian and indeed equivalent to $ALE_{\mathbb{Z}_k}$. The comparison between these two forms of the same metrics is very useful in order to get cues about the mechanisms by means of which the moment map parameters blowup the singularities in the purely Kählerian case. Hence let us start with the general form of the GH-metrics.

Let the x,y,z be the real coordinates of $\mathbb{R}^3$ to which we adjoin an angle τ spanning a circle $\mathbb{S}^1$. A general GH-metric has the following form:

$$ds^2_{\mathrm{GH}} = \frac{(\mathrm{d}\tau + \omega)^2}{\mathscr{V}} + \mathscr{V}\left(\mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2\right) \tag{C.1}$$

where $\mathscr{V} = \mathscr{V}(x,y,z)$ is a harmonic function on $\mathbb{R}^3$:

$$\frac{\partial^2 \mathscr{V}}{\partial x^2} + \frac{\partial^2 \mathscr{V}}{\partial y^2} + \frac{\partial^2 \mathscr{V}}{\partial z^2} = 0 \tag{C.2}$$

and

$$\omega = \omega_x\, dx + \omega_y\, dy + \omega_z\, dz \tag{C.3}$$

is a one-form whose external derivative is requested to be Hodge dual, in the flat metric $ds^2_{\mathbb{R}^3} = \mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2$ of $\mathbb{R}^3$, to the gradient of $\mathscr{V}$:

$$\star_{\mathbb{R}^3} \mathrm{d}\omega = \mathrm{d}\mathscr{V} \tag{C.4}$$

Without loss of generality we can choose an axial gauge for the connection ω by setting:

$$\omega_z = 0 \tag{C.5}$$

The four-dimensional Riemannian space $\mathscr{M}_{\mathrm{GH}}$, whose metric is provided by eq. (C.1), is a $\mathrm{U}(1)$-bundle over $\mathbb{R}^3$. Actually we can easily prove that $\mathscr{M}_{\mathrm{GH}}$ is Kählerian by means of the following argument. Consider the following two-form:

$$\mathbb{K}_{\mathrm{GH}} = 2\left((\mathrm{d}\tau + \omega)\wedge \mathrm{d}z - \mathscr{V}\,\mathrm{d}x \wedge \mathrm{d}y\right) \tag{C.6}$$

which is closed in force of eqs.(C.2) and (C.3):

$$\mathrm{d}\mathbb{K}_{\mathrm{GH}} = 0 \tag{C.7}$$

From eq. (C.1) we easily workout the components of the metric in the x,y,z,τ coordinate basis:

$$g_{ij} = \begin{pmatrix} \mathscr{V} + \frac{\omega_x^2}{\mathscr{V}} & \frac{\omega_x\omega_y}{\mathscr{V}} & 0 & \frac{\omega_x}{\mathscr{V}} \\ \frac{\omega_x\omega_y}{\mathscr{V}} & \mathscr{V} + \frac{\omega_y^2}{\mathscr{V}} & 0 & \frac{\omega_y}{\mathscr{V}} \\ 0 & 0 & \mathscr{V} & 0 \\ \frac{\omega_x}{\mathscr{V}} & \frac{\omega_y}{\mathscr{V}} & 0 & \frac{1}{\mathscr{V}} \end{pmatrix} \tag{C.8}$$

and of its inverse:

$$g^{ij} = \begin{pmatrix} \frac{1}{\mathscr{V}} & 0 & 0 & -\frac{\omega_x}{\mathscr{V}} \\ 0 & \frac{1}{\mathscr{V}} & 0 & -\frac{\omega_y}{\mathscr{V}} \\ 0 & 0 & \frac{1}{\mathscr{V}} & 0 \\ -\frac{\omega_x}{\mathscr{V}} & -\frac{\omega_y}{\mathscr{V}} & 0 & \frac{\mathscr{V}^2+\omega_x^2+\omega_y^2}{\mathscr{V}} \end{pmatrix} \tag{C.9}$$

Similarly, from eq. (C.6) we work out the components of the form $\mathbb{K}_{\mathrm{GH}}$:

$$K_{ij} = \begin{pmatrix} 0 & -\mathscr{V} & \omega_x & 0 \\ \mathscr{V} & 0 & \omega_y & 0 \\ -\omega_x & -\omega_y & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{C.10}$$

Raising the second index of the antisymmetric tensor K_{ij} with the inverse metric $g^{j\ell}$ we obtain a mixed tensor

$$J_i^{\ \ell} \equiv K_{ij}\, g^{j\ell} = \begin{pmatrix} 0 & -1 & \frac{\omega_x}{\mathscr{V}} & \omega_y \\ 1 & 0 & \frac{\omega_y}{\mathscr{V}} & -\omega_x \\ 0 & 0 & 0 & -\mathscr{V} \\ 0 & 0 & \frac{1}{\mathscr{V}} & 0 \end{pmatrix} \tag{C.11}$$

which satisfies the property:

$$J_i^{\ \ell} J_\ell^{\ \mathrm{m}} = -\,\delta_i^{\mathrm{m}} \tag{C.12}$$

Hence J is a *almost complex structure* which is proved to be a *complex structure* by verifying that its Nijenhuis tensor vanishes:

$$N_{ij}^{\ell} \equiv \partial_{[i} J_{j]}^{\ \ \ell} - J_i^{\ \mathrm{m}} J_j^{\ \mathrm{n}}\, \partial_{[\mathrm{m}} J_{\mathrm{n}]}^{\ \ \ \ell} = 0 \tag{C.13}$$

It follows that $\mathscr{M}_{\mathrm{GH}}$ is a complex manifold, the metric (C.6) being hermitian with respect to J since the matrix $K_{ij} \equiv J_i^{\ \ell} g_{\ell j}$ is by construction antisymmetric and, as such, it defines a Kähler 2-form. Thus we have a Kähler form which is closed and this, by definition, implies that the complex manifold $\mathscr{M}_{\mathrm{GH}}$ is a Kähler manifold.

C.1 Integration of the complex structure and the issue of the Kähler potential

The first task to put the Kähler metric of a $2n$-dimensional real manifold into a standard complex form derived from a Kähler potential is that of deriving a suitable set of complex coordinates Z_μ that are eigenstates of the complex structure. This means to find a complete set of n complex solutions of the following differential equation:

$$J_i^{\ \ell}\, \partial_\ell Z = i\, \partial_i Z \tag{C.14}$$

In the case of the complex structure in equation (C.11) a basis of the eigenspace pertaining to the eigenvalue i is easily provided by the following two vectors

$$\begin{aligned} \mathbf{v}_1 &= \{-i\omega_y, i\omega_x, i\mathscr{V}, 1\} \\ \mathbf{v}_2 &= \{i, 1, 0, 0\} \\ J\,\mathbf{v}_{1,2} &= i\mathbf{v}_{1,2} \end{aligned} \tag{C.15}$$

The second eigenvector $\mathbf{v}_2$ inserted into equation (C.14) immediately singles out one of the two complex coordinates:

$$\mathfrak{z} \equiv y + i x \tag{C.16}$$

In order to integrate eq. (C.14) utilizing the first eigenvector $\mathbf{v}_1$, a very useful tool is provided by a recent observation made by Ortin et al in [58] who pointed out that a convenient way of automatically realizing conditions (C.2) and (C.4) is obtained by setting:

$$\omega_x = \frac{\partial^2 \mathscr{F}}{\partial y \partial z} \quad ; \quad \omega_y = -\frac{\partial^2 \mathscr{F}}{\partial x \partial z} \quad ; \quad \mathscr{V} = \frac{\partial^2 \mathscr{F}}{\partial z^2} \tag{C.17}$$

where $\mathscr{F}(x,y,z)$ is a harmonic prepotential:

$$\frac{\partial^2 \mathscr{F}}{\partial x^2} + \frac{\partial^2 \mathscr{F}}{\partial y^2} + \frac{\partial^2 \mathscr{F}}{\partial z^2} = 0 \tag{C.18}$$

Using the prepotential $\mathscr{F}$ the differential equation to be satisfied by the searched for complex coordinate $\mathfrak{w}$ is the following one:

$$\left\{ i\frac{\partial^2 \mathscr{F}}{\partial z \partial z}, i\frac{\partial^2 \mathscr{F}}{\partial y \partial z}, i\frac{\partial^2 \mathscr{F}}{\partial z^2}, 1 \right\} = \{\partial_x \mathfrak{w}, \partial_y \mathfrak{w}, \partial_z \mathfrak{w}, \partial_\tau \mathfrak{w}\} \tag{C.19}$$

In view of eq. (C.17) we can set:

$$\mathscr{F}(x,y,z) = \int \mathrm{d}z \int \mathrm{d}z \, \mathscr{V}(x,y,z) \tag{C.20}$$

and the differential equation (C.19) is immediately integrated by setting:

$$\mathfrak{w} = \tau + i\,\partial_z \mathscr{F} = \tau + i \int \mathscr{V}\,\mathrm{d}z \tag{C.21}$$

Obviously, whenever a complex coordinate has been found, any invertible holomorphic function of the same is an equally good complex coordinate. Hence in addition to $\mathfrak{z}$, defined in eq. (C.16), we choose the second complex coordinate as follows:

$$\mathfrak{h} = \exp[i\mathfrak{w}] = e^{i\tau}\rho \quad ; \quad \rho = \exp\left[-\int \mathscr{V}\,\mathrm{d}z\right] \tag{C.22}$$

Using the above implicit definition of the complex coordinates one can transform the Kähler 2-form (C.6) to the complex coordinates obtaining:

$$\mathbb{K}_{\mathrm{GH}} = K_{\mathfrak{h}\bar{\mathfrak{h}}}\, d\mathfrak{h} \wedge d\bar{\mathfrak{h}} + K_{\mathfrak{h}\bar{\mathfrak{z}}}\, d\mathfrak{h} \wedge d\bar{\mathfrak{z}} + K_{\mathfrak{z}\bar{\mathfrak{h}}}\, d\mathfrak{z} \wedge d\bar{\mathfrak{h}} + K_{\mathfrak{z}\bar{\mathfrak{z}}}\, d\mathfrak{z} \wedge d\bar{\mathfrak{z}} \tag{C.23}$$

where

$$\begin{aligned}
K_{\mathfrak{h}\bar{\mathfrak{h}}} &= i\frac{1}{\mathfrak{h}\,\bar{\mathfrak{h}}\,\mathscr{V}} = \partial_{\mathfrak{h}}\bar{\partial}_{\bar{\mathfrak{h}}}\mathscr{K} \\
K_{\mathfrak{h}\bar{\mathfrak{z}}} &= \frac{i\omega_x + \omega_y}{\mathfrak{h}\,\mathscr{V}} = \partial_{\mathfrak{h}}\bar{\partial}_{\bar{\mathfrak{z}}}\mathscr{K} \\
K_{\mathfrak{z}\bar{\mathfrak{h}}} &= \frac{i\omega_x - \omega_y}{\bar{\mathfrak{h}}\,\mathscr{V}} = \partial_{\mathfrak{z}}\bar{\partial}_{\bar{\mathfrak{h}}}\mathscr{K} \\
K_{\mathfrak{z}\bar{\mathfrak{z}}} &= i\frac{\omega_x^2 + \omega_y^2 + \mathscr{V}^2}{\mathscr{V}} = \partial_{\mathfrak{z}}\bar{\partial}_{\bar{\mathfrak{z}}}\mathscr{K}
\end{aligned} \tag{C.24}$$

The problem of deriving the Kähler potential $\mathscr{K}(\mathfrak{h},\mathfrak{z},\bar{\mathfrak{h}},\bar{\mathfrak{z}})$ corresponding to the GH-metric is reduced to the nontrivial task of inverting the coordinate transformation encoded in eqs. (C.21) and (C.16) and then solving the system of coupled differential equations encoded in eqs. (C.24). Typically this is far from being a nontrivial task, but in some simple cases it can be done. The primary illuminating example is provided by the Eguchi-Hanson metric corresponding to $ALE_{\mathbb{Z}_2}$.

D Abelian Examples of generalized Kronheimer constructions

D.1 Construction à la Kronheimer of the crepant resolution $\mathscr{M}_{\mathbb{Z}_3} \to \frac{\mathbb{C}^3}{\mathbb{Z}_3}$ with $\mathbb{Z}_3 \subset \mathrm{L}_{168}$

Our next exercise is the resolution of the singularity $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$ where the group $\mathbb{Z}_3$ is embedded in $G_{21} \subset L_{168}$. We will see that here the algebraic moment map equations are of higher order, actually reduce to a single equation of the sixth order which cannot be explicitly solved and no analytically close expression of either the Kähler potential or the first Chern classes of the tautological bundles can be written down. Yet from another view point we know that the resolved manifold is the product $\mathbb{C} \times ALE_{\mathbb{Z}_3}$, the second factor being equivalent to a Gibbons-Hawking space for which an explicit expression of the metric exists although written in different coordinates.

Following the general strategy outlined in the main text sections, the first step consists in deriving the invariant space $\mathscr{S}_\Gamma = \mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R)$ made of triples $\{A, B, C\}$ of 3×3 matrices that satisfy eq. (6.6), namely:

$$\mathrm{X}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} R(\mathrm{X}^{-1})\,A\,R(\mathrm{X}) \\ R(\mathrm{X}^{-1})\,B\,R(\mathrm{X}) \\ R(\mathrm{X}^{-1})\,C\,R(\mathrm{X}) \end{pmatrix} \tag{D.1}$$

where X is the $\mathbb{Z}_3$-generator displayed in eq (A.7) and $R(\mathrm{X})$ is its representation in the natural basis of the regular representation. Quite exceptionally this latter coincides with X as a matrix:

$$R(\mathrm{X}) = \mathrm{X} \tag{D.2}$$

The constraint (D.1) reduces the number of parameters from 27 to 9. Explicitly we have:

$$\begin{array}{rcl} (A,B,C) & \in & \mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R) \\ & \Downarrow & \\ A & = & \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \\ B & = & \begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,1} \\ \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,1} \\ \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,1} \end{pmatrix} \\ C & = & \begin{pmatrix} \alpha_{3,3} & \alpha_{3,1} & \alpha_{3,2} \\ \alpha_{1,3} & \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,3} & \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \end{array} \tag{D.3}$$

If we perform the diagonalization of the regular representation by means of the matrix (A.14), the result of the same constraint (D.1) can be expressed in the following way:

$$\begin{aligned}
(A,B,C) &\in \mathrm{Hom}_\Gamma(R,\mathscr{Q}\otimes R)\\
&\Downarrow\\
A &= \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3}\\ m_{2,1} & m_{2,2} & m_{2,3}\\ m_{3,1} & m_{3,2} & m_{3,3}\end{pmatrix}\\
B &= \begin{pmatrix} m_{1,1} & -\sqrt[3]{-1}m_{1,2} & (-1)^{2/3}m_{1,3}\\ (-1)^{2/3}m_{2,1} & m_{2,2} & -\sqrt[3]{-1}m_{2,3}\\ -\sqrt[3]{-1}m_{3,1} & (-1)^{2/3}m_{3,2} & m_{3,3}\end{pmatrix}\\
C &= \begin{pmatrix} m_{1,1} & (-1)^{2/3}m_{1,2} & -\sqrt[3]{-1}m_{1,3}\\ -\sqrt[3]{-1}m_{2,1} & m_{2,2} & (-1)^{2/3}m_{2,3}\\ (-1)^{2/3}m_{3,1} & -\sqrt[3]{-1}m_{3,2} & m_{3,3}\end{pmatrix}
\end{aligned} \tag{D.4}$$

in terms of a new set of nine parameters $m_{i,j}$.

D.1.1 Characterization of the locus $\mathscr{D}_{\mathbb{Z}_3}$ and of the variety $\mathbb{V}_{3+2}$

Next we define the reduction of the invariant space $\mathrm{Hom}_\Gamma(R,\mathscr{Q}\otimes R)$ to the locus cut out by the holomorphic constraint (6.12) which was named $\mathscr{D}_\Gamma$:

$$\mathscr{D}_{\mathbb{Z}_3} \equiv \left\{ p = \begin{pmatrix} A\\ B\\ C\end{pmatrix} \in \mathrm{Hom}_\Gamma(R,\mathscr{Q}\otimes R) \;/\; [A,B]=[B,C]=[C,A]=0 \right\} \tag{D.5}$$

Differently from the more complicated cases of maximal subgroups of L_{168}, in the present abelian case we can explicitly solve the quadratic equations provided by the commutator constraints and we discover that there is a principal branch of the solution, named $\mathscr{D}^0_\Gamma$ that has indeed dimension $5=|\Gamma|+2$. However, in addition to that there are 25 more branches of the solution with smaller dimension. This clarifies a point stated by us in our previous general discussion. There is always a unique principal branch of the solution with the maximal number $|\Gamma|+2$ of free parameters and we are able to show that such principal branch $\mathscr{D}^0_\Gamma$ is indeed the orbit with respect to the group $\mathscr{G}_\Gamma$ of the subspace L_Γ defined in eq. (5.34). Hence the variety $\mathbb{V}_{|\Gamma|+2}$ of which we are supposed to take the Kähler quotient is not defined by eq. (D.5), rather by the principal branch of that variety.

In the present case, the principal branch of the solution to eq. (D.5) can be expressed in the following way:

$$m_{1,1}=\omega_1,\, m_{1,2}=\omega_2,\, m_{1,3}=\omega_3,\, m_{2,1}=\omega_4,\, m_{2,2}=\omega_1,\, m_{2,3}=\omega_5,\, m_{3,1}=\frac{\omega_2\omega_4}{\omega_3},\, m_{3,2}=\frac{\omega_2\omega_4}{\omega_5},\, m_{3,3}=\omega_1 \tag{D.6}$$

where $\omega_{1,\dots,5}$ are five free complex parameters. Substituting eq. (D.6) in eq. (D.4) we obtain the explicit parameterization of the locus $\mathscr{D}^0_{\mathbb{Z}_3}$ in terms of the 5 parameters ω_i.

D.1.2 Derivation of the quiver group $\mathscr{G}_\Gamma$

Our next point is the derivation of the group $\mathscr{G}_\Gamma$ defined in eq. (6.19), namely:

$$\mathscr{G}_{\mathbb{Z}_3} = \{ g\in \mathrm{SL}(3,\mathbb{C}) \quad | \quad \forall\gamma\in\mathbb{Z}_3 :\; [D_R(\gamma),D_{\mathrm{def}}(g)]=0\} \tag{D.7}$$

Let us proceed to this construction. In the diagonal basis of the regular representation this is a very easy task, since the group is simply given by the diagonal 3×3 matrices with determinant one. We introduce these matrices:

$$g = \left(\begin{array}{|c|c|c|}\hline a_1 & 0 & 0 \\ \hline 0 & a_2 & 0 \\ \hline 0 & 0 & a_3 \\ \hline\end{array}\right) \tag{D.8}$$

subject to the constraint $\prod_{i=1}^{3} a_i$=1.

Next we want to show that

$$\mathscr{D}^0_{\mathbb{Z}_3} = \mathrm{Orbit}_{\mathscr{G}_{\mathbb{Z}_3}}\left(L_{\mathbb{Z}_3}\right) \tag{D.9}$$

To this effect we rewrite the locus

$$\begin{aligned} L_{\mathbb{Z}_3} &\equiv \{A_0, B_0, C_0\} \\ A_0 &= \begin{pmatrix} \alpha_{1,1} & 0 & 0 \\ 0 & \alpha_{2,2} & 0 \\ 0 & 0 & \alpha_{3,3} \end{pmatrix} ; B_0 = \begin{pmatrix} \alpha_{2,2} & 0 & 0 \\ 0 & \alpha_{3,3} & 0 \\ 0 & 0 & \alpha_{1,1} \end{pmatrix} ; C_0 = \begin{pmatrix} \alpha_{3,3} & 0 & 0 \\ 0 & \alpha_{1,1} & 0 \\ 0 & 0 & \alpha_{2,2} \end{pmatrix} \end{aligned} \tag{D.10}$$

in the diagonal basis of the regular representation. The change of basis is performed by the normalized character table of the cyclic group $\mathbb{Z}_3$

$$\boldsymbol{\chi} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix} \tag{D.11}$$

by setting $\widetilde{A}_0 = \boldsymbol{\chi}^\dagger A_0 \boldsymbol{\chi}$ and similarly for $\widetilde{B}_0$ and $\widetilde{C}_0$. Next let us setup the orbit of interest to us:

$$\mathrm{Orbit}_{\mathscr{G}_{\mathbb{Z}_3}}\left(L_{\mathbb{Z}_3}\right) \equiv \left\{g^{-1}\boldsymbol{\chi}^\dagger A_0 \boldsymbol{\chi}\, g,\, g^{-1}\boldsymbol{\chi}^\dagger B_0 \boldsymbol{\chi}\, g,\, g^{-1}\boldsymbol{\chi}^\dagger C_0 \boldsymbol{\chi}\, g\right\} \tag{D.12}$$

where g is given in eq. (D.8). The space (D.12) is clearly 5-dimensional, the five parameters being, $\alpha_{1,1}, \alpha_{2,2}, \alpha_{3,3}$ and a_1, a_2, a_3 with constraint $\prod_{i=1}^{3} a_i = 1$. We can easily verify that the identification (D.9) holds true upon use of the following identification:

$$
\begin{aligned}
a_1 &= \frac{\omega_4^{1/3}}{\omega_3^{1/3}} \\
a_2 &= \frac{\omega_2^{1/3}}{\omega_5^{1/3}} \\
a_3 &= \frac{\omega_3^{1/3}\omega_5^{1/3}}{\omega_2^{1/3}\omega_4^{1/3}} \\
\alpha_{1,1} &= \omega_1 + \frac{\omega_2^{1/3}\omega_3^{1/3}\omega_4^{2/3}}{\omega_5^{1/3}} + \frac{\omega_2^{2/3}\omega_4^{1/3}\omega_5^{1/3}}{\omega_3^{1/3}} \\
\alpha_{2,2} &= \frac{1}{6}\left(6\omega_1 + \frac{3i\left(i+\sqrt{3}\right)\omega_2^{1/3}\omega_3^{1/3}\omega_4^{2/3}}{\omega_5^{1/3}} - \frac{6(-1)^{1/3}\omega_2^{2/3}\omega_4^{1/3}\omega_5^{1/3}}{\omega_3^{1/3}}\right) \\
\alpha_{3,3} &= \frac{-2\sqrt{3}\omega_2^{1/3}\omega_3^{2/3}\omega_4^{2/3} + \left(-3i+\sqrt{3}\right)\omega_1\omega_3^{1/3}\omega_5^{1/3} + \left(3i+\sqrt{3}\right)\omega_2^{2/3}\omega_4^{1/3}\omega_5^{2/3}}{\left(-3i+\sqrt{3}\right)\omega_3^{1/3}\omega_5^{1/3}}
\end{aligned}
\tag{D.13}
$$

between the orbit parameters and those that parameterize, according to (D.6), the principal branch $\mathscr{D}^0_{\mathbb{Z}_3}$ of the locus (6.12).

In conclusion the variety $\mathbb{V}_5$ of which we are supposed to take the Kähler quotient in order to obtain the resolution $\mathscr{M}_\zeta \to \frac{\mathbb{C}^3}{\mathbb{Z}_3}$ is defined in the following way:

$$
\mathbb{V}_5 \equiv \mathrm{Orbit}_{\mathscr{G}_{\mathbb{Z}_3}} = \{A, B, C\} \tag{D.14}
$$

$$
\begin{aligned}
A &= \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_4 & \omega_1 & \omega_5 \\ \frac{\omega_2\omega_4}{\omega_3} & \frac{\omega_2\omega_4}{\omega_5} & \omega_1 \end{pmatrix} \\
B &= \begin{pmatrix} \omega_1 & -\sqrt[3]{-1}\omega_2 & (-1)^{2/3}\omega_3 \\ (-1)^{2/3}\omega_4 & \omega_1 & -\sqrt[3]{-1}\omega_5 \\ -\frac{\sqrt[3]{-1}\omega_2\omega_4}{\omega_3} & \frac{(-1)^{2/3}\omega_2\omega_4}{\omega_5} & \omega_1 \end{pmatrix} \\
C &= \begin{pmatrix} \omega_1 & (-1)^{2/3}\omega_2 & -\sqrt[3]{-1}\omega_3 \\ -\sqrt[3]{-1}\omega_4 & \omega_1 & (-1)^{2/3}\omega_5 \\ \frac{(-1)^{2/3}\omega_2\omega_4}{\omega_3} & -\frac{\sqrt[3]{-1}\omega_2\omega_4}{\omega_5} & \omega_1 \end{pmatrix}
\end{aligned}
\tag{D.15}
$$

D.1.3 The algebraic equation of the orbifold locus

Let us now consider the action of the $\mathbb{Z}_3$ group on $\mathbb{C}^3$ as defined by the generator X in eq. (A.7). If $\{z_1, z_2, z_3\}$ are the coordinates of a point in $\mathbb{C}^3$, we see that there are four invariant polynomials:

$$
\begin{aligned}
J_1 &= \frac{1}{216}\left(2\sqrt{3}z_1 - \left(\sqrt{3}+3i\right)z_2 - \left(\sqrt{3}-3i\right)z_3\right)^3 \equiv w_1^3 \\
J_2 &= \frac{1}{216}\left(2\sqrt{3}z_1 - \left(\sqrt{3}-3i\right)z_2 - \left(\sqrt{3}+3i\right)z_3\right)^3 \equiv w_2^3 \\
J_3 &= \frac{(z_1+z_2+z_3)^3}{3\sqrt{3}} \equiv w_3^3 \\
J_4 &= \frac{z_1^3 - 3z_2z_3z_1 + z_2^3 + z_3^3}{3\sqrt{3}} \equiv w_1\,w_2\,w_3
\end{aligned}
\tag{D.16}
$$

which satisfy the following equation:

$$
J_1\,J_2\,J_3 - J_4^3 = 0 \tag{D.17}
$$

This equation can be regarded as the cubic equation which cuts out in $\mathbb{C}^4$ the locus corresponding to the singular orbifold $\mathbb{C}^3/\mathbb{Z}_3$. As we know this latter is equivalent to $\mathbb{C}\times\mathbb{C}^3/\mathbb{Z}_3$. How do we retrieve this fact in the present language? It is a simple matter. Consider the new coordinates of $\mathbb{C}^3$ that diagonalize the action of X and are implicitly defined already in eq. (D.16):

$$
\begin{aligned}
\mathbf{w} &\equiv \boldsymbol{\chi}^\dagger\,\mathbf{z} \\
w_1 &= \tfrac{1}{6}\left(2\sqrt{3}z_1 - \left(\sqrt{3}+3i\right)z_2 - \left(\sqrt{3}-3i\right)z_3\right) \\
w_2 &= \tfrac{1}{6}\left(2\sqrt{3}z_1 - \left(\sqrt{3}-3i\right)z_2 - \left(\sqrt{3}+3i\right)z_3\right) \\
w_3 &= \tfrac{1}{\sqrt{3}}\left(z_1+z_2+z_3\right)
\end{aligned}
\tag{D.18}
$$

We have the correspondence:

$$
\mathrm{X}\,\mathbf{z} \Leftrightarrow \widetilde{\mathrm{X}}\,\mathbf{w} \quad \text{where} \quad \widetilde{\mathrm{X}} = \begin{pmatrix} e^{\frac{2\pi}{3}i} & 0 & 0 \\ 0 & e^{-\frac{2\pi}{3}i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{D.19}
$$

which implies that in terms of the w-variables we have the following $\mathbb{Z}_3$ invariants:

$$
\mathrm{j}_1 = w_1^3 \quad ; \quad \mathrm{j}_2 = w_2^3 \quad ; \quad \mathrm{j}_3 = w_1\,w_2; \quad ; \quad \mathrm{j}_4 = w_3 \tag{D.20}
$$

and we can write:

$$
J_1 = \mathrm{j}_1 \quad ; \quad J_2 = \mathrm{j}_2 \quad ; \quad J_3 = \mathrm{j}_4^3 \quad ; \quad J_4 = \mathrm{j}_3\,\mathrm{j}_4 \tag{D.21}
$$

Regarding J_i and j_i as the coordinates of two copies of $\mathbb{C}^4$ we can regard eq. (D.21) as a morphism:

$$
\mu\,:\,\mathbb{C}^3\times\mathbb{C}\to\mathbb{C}^4 \tag{D.22}
$$

Under such a morphism the algebraic equation (D.17) is mapped into:

$$
\mathrm{j}_4^3\left(\mathrm{j}_1\,\mathrm{j}_2 - \mathrm{j}_3^3\right) = 0 \tag{D.23}
$$

and in the expression in bracket we recognize the equation of the $\frac{\mathbb{C}^2}{\mathbb{Z}_3}$ orbifold as described in eq. (5.38) while discussing the standard Kronheimer construction (it suffices to identify $x = \mathfrak{j}_1, y = \mathfrak{j}_2, z = \mathfrak{j}_3$).

D.1.4 Map of the variety $\mathbb{V}_5$ into the algebraic locus corresponding to the orbifold

Having established the above relations we verify that, in complete analogy with the standard Kronheimer construction, we can reproduce the defining equation (D.17) in terms of invariants of the three matrices (D.15) spanning the $\mathbb{V}_5$ variety. It suffices to identify:

$$\begin{aligned}
J_1 &= \mathrm{Det}[A] = \omega_1^3 - 3\omega_2\omega_4\omega_1 + \frac{\omega_2^2\omega_4\omega_5}{\omega_3} + \frac{\omega_2\omega_3\omega_4^2}{\omega_5} \\
J_2 &= \mathrm{Det}[B] = \omega_1^3 - 3\omega_2\omega_4\omega_1 + \frac{\omega_2^2\omega_4\omega_5}{\omega_3} + \frac{\omega_2\omega_3\omega_4^2}{\omega_5} \\
J_3 &= \mathrm{Det}[C] = \omega_1^3 - 3\omega_2\omega_4\omega_1 + \frac{\omega_2^2\omega_4\omega_5}{\omega_3} + \frac{\omega_2\omega_3\omega_4^2}{\omega_5} \\
J_4 &= \mathrm{Tr}[ABC] = \left(\omega_1^3 - 3\omega_2\omega_4\omega_1 + \frac{\omega_2^2\omega_4\omega_5}{\omega_3} + \frac{\omega_2\omega_3\omega_4^2}{\omega_5}\right)^3
\end{aligned} \tag{D.24}$$

Eq. (D.24) describes an explicit inclusion map of the variety $\mathbb{V}_5$ into the algebraic locus $\mathbb{C}^3/\mathbb{Z}_3$:

$$\mathbb{V}_5 \to \frac{\mathbb{C}^3}{\mathbb{Z}_3} \tag{D.25}$$

D.1.5 The Kähler quotient

The next step consists in performing the Kähler quotient of the Kähler manifold $\mathbb{V}_5$ with respect to the compact subgroup of the quiver group $\mathscr{G}_{\mathbb{Z}_3}$, which, as we several times emphasized, is the *gauge group* of the corresponding three-dimensional Chern-Simons gauge theory:

$$\mathscr{F}_{\mathbb{Z}_3} \equiv \mathscr{G}_{\mathbb{Z}_3} \bigcap \mathrm{SU}(3) \tag{D.26}$$

A generic element $g \in \mathscr{F}_{\mathbb{Z}_3}$ is of the form (D.8) with:

$$a_i = \exp[i\,\theta_i] \quad ; \quad \sum_{i=1}^{3} \theta_i = 2\pi n \quad (n \in \mathbb{Z}) \tag{D.27}$$

The Kähler structure of $\mathbb{V}_5$ is provided by the pullback on the $\mathbb{V}_5$ surface of the Kähler potential of the entire flat Kähler manifold $\mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q} \otimes R)$, namely we have:

$$\begin{aligned}
\mathscr{K}_{\mathbb{V}} &\equiv \mathrm{Tr}\left(\left[A^\dagger, A\right] + \left[B^\dagger, B\right] + \left[C^\dagger, C\right]\right) \\
&= 3\left(3\omega_1\overline{\omega}_1 + \omega_3\overline{\omega}_3 + \omega_4\overline{\omega}_4 + \omega_2\overline{\omega}_2\left(\frac{\omega_4\overline{\omega}_4}{\omega_3\overline{\omega}_3} + \frac{\omega_4\overline{\omega}_4}{\omega_5\overline{\omega}_5} + 1\right) + \omega_5\overline{\omega}_5\right)
\end{aligned} \tag{D.28}$$

$\mathscr{K}_{\mathbb{V}}$ is obviously invariant under the unitary transformations of the *gauge group* :

$$\forall g \in \mathscr{F}_{\mathbb{Z}_3} : \quad \{A, B, C\} \to \left\{g^\dagger A g, g^\dagger B g, g^\dagger C g\right\} \tag{D.29}$$

which, for that reason, is an *isometry group* of the corresponding Kähler metric on $\mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q} \otimes R)$ and of its restriction to $\mathbb{V}_5$. The last point follows from the fact that, by construction, $\mathscr{F}_{\mathbb{Z}_3}$ maps $\mathbb{V}_5 \subset \mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q} \otimes R)$ into itself.

A basis of two linearly independent generators of the Lie algebra $\mathbb{F}_{\mathbb{Z}_3}$ is provided by the following two matrices:

$$\mathfrak{f}_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \mathfrak{f}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \tag{D.30}$$

and the moment maps corresponding to the two isometries generated by them are defined as follows:

$$\mathfrak{P}_A = -i\,\mathrm{Tr}\left(\mathfrak{f}_A\left(\left[A^\dagger, A\right] + \left[B^\dagger, B\right] + \left[C^\dagger, C\right]\right)\right), \quad (A = 1,2) \tag{D.31}$$

Explicitly we find:

$$\begin{aligned} \mathfrak{P}_1(\omega,\overline{\omega}) &= 3\omega_3\overline{\omega}_3 - 6\omega_4\overline{\omega}_4 + 3\omega_2\overline{\omega}_2\left(-\frac{\omega_4\overline{\omega}_4}{\omega_3\overline{\omega}_3} + \frac{\omega_4\overline{\omega}_4}{\omega_5\overline{\omega}_5} + 2\right) - 3\omega_5\overline{\omega}_5 \\ \mathfrak{P}_2(\omega,\overline{\omega}) &= 3\left(\omega_3\overline{\omega}_3 + \omega_4\overline{\omega}_4 + \omega_2\overline{\omega}_2\left(-\frac{\omega_4\overline{\omega}_4}{\omega_3\overline{\omega}_3} - \frac{2\omega_4\overline{\omega}_4}{\omega_5\overline{\omega}_5} - 1\right) + 2\omega_5\overline{\omega}_5\right) \end{aligned} \tag{D.32}$$

Choosing a gauge. As we emphasized several times, the quiver group $\mathscr{G}_{\mathbb{Z}_3}$ leaves the surface $[A,B] = [B,C] = [C,A] = 0$ invariant, namely it maps V_5 into itself, yet it does not leave the moment maps $\mathfrak{P}_A$ invariant since it is not an isometry. The latter are invariant only under the compact gauge subgroup. This property is very important since it is the key instrument to obtain the lifting from a zero level to a prescribed one of the level surfaces, thus providing the algorithm to perform the Kähler quotient explicitly.

To this effect we consider the action of an element:

$$g = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \frac{\mu_2}{\mu_1} & 0 \\ 0 & 0 & \frac{1}{\mu_2} \end{pmatrix} \in \mathscr{G}_{\mathbb{Z}_3} \quad ; \quad \mu_{1,2} \in \mathbb{C} \tag{D.33}$$

on the coordinates ω_i. Such an action is easily worked out to be the following one:

$$\left\{\omega_1 \to \omega_1, \omega_2 \to \frac{\mu_2\omega_2}{\mu_1^2}, \omega_3 \to \frac{\omega_3}{\mu_1\mu_2}, \omega_4 \to \frac{\mu_1^2\omega_4}{\mu_2}, \omega_5 \to \frac{\mu_1\omega_5}{\mu_2^2}\right\} \tag{D.34}$$

Relying on this we can introduce three complex coordinates u_i $(i = 1,2,3)$ parameterizing the locus $L_{\mathbb{Z}_3}$ and we identify the remaining two complex coordinates as parameters of the quiver group. With some ingenuity we have singled out the following transformation:

$$\omega_1 \to u_1\,, \quad \omega_2 \to \frac{\mu_2 u_2^2}{\mu_1^2 u_3}\,, \quad \omega_3 \to \frac{u_3^2}{\mu_1\mu_2 u_2}\,, \quad \omega_4 \to \frac{\mu_1^2 u_3^2}{\mu_2 u_2}\,, \quad \omega_5 \to \frac{\mu_1 u_3^2}{\mu_2^2 u_2} \tag{D.35}$$

If we separate the modulus from the phase of the complex parameters $\mu_{1,2}$ by writing:

$$\mu_{1,2} = \Omega_{1,2}\,e^{i\theta_{1,2}} \quad \Omega_{1,2} \in \mathbb{R}_+ \tag{D.36}$$

the substitution (D.35) can be rewritten as follows:

$$\omega_1 \to u_1\,, \quad \omega_2 \to \frac{e^{2\theta_2-2\theta_1}u_2^2\Omega_2}{u_3\Omega_1^2}\,, \quad \omega_3 \to \frac{e^{-\theta_1-2\theta_2}u_3^2}{u_2\Omega_1\Omega_2}\,, \quad \omega_4 \to \frac{e^{2\theta_1-2\theta_2}u_3^2\Omega_1^2}{u_2\Omega_2}\,, \quad \omega_5 \to \frac{e^{\theta_1-4\theta_2}u_3^2\Omega_1}{u_2\Omega_2^2} \tag{D.37}$$

which can be implemented into eqs. (D.32) that define the moment maps. One easily verifies that the phases

θ_i disappear in the resulting expressions as a result of the invariance of the moment maps with respect to the gauge-group. Neither do the moment maps depend from the phases of the complex coordinates u_i. Setting

$$u_{1,2,3} = \sqrt{\Delta_{1,2,3}}\, e^{i\phi_{1,2,3}} \quad \Delta_{1,2,3} \in \mathbb{R}_+ \tag{D.38}$$

and using the following convenient new basis for the gauge Lie algebra and for the moment maps:

$$\widehat{\mathfrak{P}}_1 = \tfrac{1}{3}(2\mathfrak{P}_1 + \mathfrak{P}_2) \quad ; \quad \widehat{\mathfrak{P}}_2 = \tfrac{1}{3}(\mathfrak{P}_1 + 2\mathfrak{P}_2) \tag{D.39}$$

we obtain the following quite simple result:

$$\left\{\widehat{\mathfrak{P}}_1\,,\,\widehat{\mathfrak{P}}_2\right\} = \left\{-\frac{3\left(\Omega_1^6-1\right)\left(\Delta_2^3\Omega_2^4+\Delta_3^3\Omega_1^2\right)}{\Delta_2\Delta_3\Omega_1^4\Omega_2^2}, \frac{3\left(\Omega_1^4+\Omega_2^2\right)\left(\Delta_3^3-\Delta_2^3\Omega_2^6\right)}{\Delta_2\Delta_3\Omega_1^2\Omega_2^4}\right\} \tag{D.40}$$

In order to study the moment map equations:

$$\left\{\widehat{\mathfrak{P}}_1\,,\,\widehat{\mathfrak{P}}_2\right\} = \{\zeta_1\,,\,\zeta_2\} \tag{D.41}$$

where $\zeta_{1,2}$ are the level parameters it is convenient to make the following change of parameterization:

$$\Omega_1 \to \sqrt[3]{\Upsilon_1}\sqrt[6]{\Upsilon_2} \quad , \quad \Omega_2 \to \sqrt[6]{\Upsilon_1}\sqrt[3]{\Upsilon_2} \tag{D.42}$$

The remaining task is that of solving eq. (D.41) as an algebraic equation for the parameters $\Upsilon_{1,2}$ to be determined in terms of the coordinates u_i and of the level parameters $\zeta_{1,2}$.

Calibration of the solutions at $\zeta_{1,2} = 0$ Upon the substitution (D.42), eq. (D.41) can be easily solved for $\zeta_{1,2} = 0$ and we find the following six solutions

1)	$\Upsilon_1 \to -1$	$\Upsilon_2 \to 1$
2)	$\Upsilon_1 \to -1$	$\Upsilon_2 \to -\frac{\Delta_3^3}{\Delta_2^3}$
3)	$\Upsilon_1 \to \frac{\Delta_2}{\Delta_3}$	$\Upsilon_2 \to \frac{\Delta_3^2}{\Delta_2^2}$
4)	$\Upsilon_1 \to -\frac{\sqrt[3]{-1}\Delta_2}{\Delta_3}$	$\Upsilon_2 \to -\frac{\sqrt[3]{-1}\Delta_3^2}{\Delta_2^2}$
5)	$\Upsilon_1 \to \frac{(-1)^{2/3}\Delta_2}{\Delta_3}$	$\Upsilon_2 \to \frac{(-1)^{2/3}\Delta_3^2}{\Delta_2^2}$
6)	$\Upsilon_1 \to \frac{\Delta_2^3}{\Delta_3^3}$	$\Upsilon_2 \to -\frac{\Delta_3^3}{\Delta_2^3}$

(D.43)

Inspecting the above result we immediately see that only the solution 3) is acceptable since it is the only one for which both Υ_1 and Υ_2 are real and positive. In terms of the original parameters $\Omega_{1,2}$ solution 3) means:

$$\Omega_1 \to 1 \quad ; \quad \Omega_2 \to \frac{|u_3|}{|u_2|} \tag{D.44}$$

and reinstalling the phases we can argue that it corresponds to the following complex transformation of the quiver group:

$$\mu_1 \to 1 \quad ; \quad \mu_2 \to \frac{u_3}{u_2} \tag{D.45}$$

Inserting eq. (D.45) in eq. (D.35) and replacing the ω_i accordingly in eq. (D.15) we obtain a set of three matrices that we name $\widehat{A}_0, \widehat{B}_0, \widehat{C}_0$. Transforming these latter back to the natural basis of the regular representation by setting $A_0 \equiv \boldsymbol{\chi} \widehat{A}_0 \boldsymbol{\chi}^\dagger$ and similarly for B_0 and C_0 we get:

$$
\begin{aligned}
A_0 &= \begin{pmatrix} u_1+u_2+u_3 & 0 & 0 \\ 0 & \frac{1}{2}\left(2u_1+\left(-1-i\sqrt{3}\right)u_2+i\left(\sqrt{3}+i\right)u_3\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(2u_1+i\left(\sqrt{3}+i\right)u_2+\left(-1-i\sqrt{3}\right)u_3\right) \end{pmatrix} \\
B_0 &= \begin{pmatrix} u_1-\sqrt[3]{-1}u_2+(-1)^{2/3}u_3 & 0 & 0 \\ 0 & \frac{1}{2}\left(2u_1+i\left(\sqrt{3}+i\right)u_2+\left(-1-i\sqrt{3}\right)u_3\right) & 0 \\ 0 & 0 & u_1+u_2+u_3 \end{pmatrix} \\
C_0 &= \begin{pmatrix} u_1+(-1)^{2/3}u_2-\sqrt[3]{-1}u_3 & 0 & 0 \\ 0 & u_1+u_2+u_3 & 0 \\ 0 & 0 & \frac{1}{2}\left(2u_1+\left(-1-i\sqrt{3}\right)u_2+i\left(\sqrt{3}+i\right)u_3\right) \end{pmatrix}
\end{aligned}
\tag{D.46}
$$

Being diagonal and belonging to $\mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q}\otimes R)$, the triple of matrices $\{A_0, B_0, C_0\}$ lies in the locus $L_{\mathbb{Z}_3}$ which we have already established to be isomorphic with the singular orbifold $\mathbb{C}^3/\mathbb{Z}_3$. The coordinates $u_{1,2,3}$ just provide a linear parameterization of $\mathbb{C}^3/\mathbb{Z}_3$. This shows that at zero-level of the momentum map we are back to the singular orbifold. The resolution of singularities occur at the non-vanishing levels.

D.1.6 Solutions of the moment map equations at $\zeta \neq 0$

When $\zeta_{1,2} \neq 0$ we have been able to find the solutions of the moment map equation (D.41) in an implicit way in terms of the roots of a sixth order equation whose coefficients depend on the level parameters $\zeta_{1,2}$ and on the moduli-square $\Delta_{1,2,3}$ of the complex coordinates $u_{1,2,3}$. The resolving sixth-order equation has the form:

$$
\sum_{n=0}^{6} c_n \alpha^n = 0 \tag{D.47}
$$

where the coefficients have the form:

$$
\begin{aligned}
c_1 &= 9\Delta_3^6 \\
c_2 &= 3\Delta_3^3\left(-3\Delta_2^3+6\Delta_3^3+\Delta_2\Delta_3\left(2\zeta_1-\zeta_2\right)\right) \\
c_3 &= \Delta_3^2\left(-18\Delta_2^3\Delta_3+9\Delta_3^4+\Delta_2^2\zeta_1\left(\zeta_1-\zeta_2\right)+6\Delta_2\Delta_3^2\left(2\zeta_1-\zeta_2\right)\right) \\
c_4 &= \Delta_2\Delta_3\left(-18\Delta_2^2\Delta_3^2+3\Delta_3^3\left(2\zeta_1-\zeta_2\right)+\Delta_2^3\left(-6\zeta_1+3\zeta_2\right)+\Delta_2\Delta_3\left(2\zeta_1^2-2\zeta_1\zeta_2+\zeta_2^2\right)\right) \\
c_5 &= \Delta_2^2\left(9\Delta_2^4-18\Delta_2\Delta_3^3+\Delta_3^2\zeta_1\left(\zeta_1-\zeta_2\right)+6\Delta_2^2\Delta_3\left(-2\zeta_1+\zeta_2\right)\right) \\
c_6 &= 3\Delta_2^3\left(6\Delta_2^3-3\Delta_3^3+\Delta_2\Delta_3\left(-2\zeta_1+\zeta_2\right)\right) \\
c_7 &= 9\Delta_2^6
\end{aligned}
\tag{D.48}
$$

Let us name α_i the 6 roots of equation (D.47). For each $\alpha = \alpha_i$, the solution of the momentum-map equations is given by:

$$
\begin{aligned}
\Upsilon_1 =& \alpha \\
\Upsilon_2 =& \\
&\left(-81(1+\alpha)\Delta_2^{11}+81\left(1+2\alpha+\alpha^2+\alpha^3+\alpha^4\right)\Delta_2^8\Delta_3^3-54\alpha^3(1+\alpha)^2\Delta_3^{10}\zeta_2-9\alpha^2(1+\alpha)^2(2+\alpha)\Delta_2\Delta_3^8\left(2\zeta_1-\zeta_2\right)\zeta_2+\right.\\
&27\Delta_2^9\Delta_3\left(\left(1+3\alpha+2\alpha^2\right)\zeta_1-\left(-2+\alpha^2\right)\zeta_2\right)-27\Delta_2^6\Delta_3^4\left(3\alpha^2\zeta_1+3\alpha^4\zeta_1+4\zeta_2+\alpha\left(3\zeta_1+\zeta_2\right)+\alpha^3\left(3\zeta_1+\zeta_2\right)\right)-\\
&9\Delta_2^7\Delta_3^2\left(3\alpha^2\zeta_1^2+\alpha^3\zeta_1\left(\zeta_1-\zeta_2\right)+\left(4\zeta_1-\zeta_2\right)\zeta_2+\alpha\left(2\zeta_1^2+5\zeta_1\zeta_2-4\zeta_2^2\right)\right)+\\
&9\Delta_2^4\Delta_3^5\left(\alpha\left(8\zeta_1-5\zeta_2\right)\zeta_2+\left(4\zeta_1-\zeta_2\right)\zeta_2+\alpha^3\left(3\zeta_1^2+6\zeta_1\zeta_2-4\zeta_2^2\right)+\alpha^2\left(\zeta_1^2+7\zeta_1\zeta_2-3\zeta_2^2\right)+\right.\\
&\left.\alpha^4\left(2\zeta_1^2+3\zeta_1\zeta_2-2\zeta_2^2\right)\right)+\\
&3\alpha\Delta_2^2\Delta_3^6\left(27\alpha^4\Delta_3^3+2\zeta_1\zeta_2\left(-\zeta_1+\zeta_2\right)+\alpha\zeta_2\left(-8\zeta_1^2+8\zeta_1\zeta_2-3\zeta_2^2\right)-2\alpha^2\zeta_2\left(5\zeta_1^2-5\zeta_1\zeta_2+\zeta_2^2\right)+\right.\\
&\left.\alpha^3\left(27\Delta_3^3-\zeta_2\left(-2\zeta_1+\zeta_2\right)^2\right)\right)-\\
&3\Delta_2^5\Delta_3^3\left(54\alpha^4\Delta_3^3+27\alpha^5\Delta_3^3+\zeta_1\zeta_2\left(-2\zeta_1+\zeta_2\right)+\alpha^3\left(27\Delta_3^3-\zeta_1^3+\zeta_1\zeta_2^2\right)+\alpha^2\left(27\Delta_3^3-\zeta_1^3-5\zeta_1^2\zeta_2+5\zeta_1\zeta_2^2-2\zeta_2^3\right)+\right.\\
&\left.\alpha\left(27\Delta_3^3+\zeta_2\left(-8\zeta_1^2+9\zeta_1\zeta_2-2\zeta_2^2\right)\right)\right)+\\
&\Delta_2^3\Delta_3^4\left(54\Delta_3^3\zeta_2+81\alpha^4\Delta_3^3\left(\zeta_1+\zeta_2\right)+27\alpha^5\Delta_3^3\left(\zeta_1+\zeta_2\right)+\alpha^2\zeta_2\left(54\Delta_3^3-4\zeta_1^3+6\zeta_1^2\zeta_2-4\zeta_1\zeta_2^2+\zeta_2^3\right)+\right.\\
&\left.\left.\alpha\zeta_2\left(54\Delta_3^3-\zeta_1\left(2\zeta_1^2-3\zeta_1\zeta_2+\zeta_2^2\right)\right)+\alpha^3\left(27\Delta_3^3\left(2\zeta_1+3\zeta_2\right)-\zeta_1\zeta_2\left(2\zeta_1^2-3\zeta_1\zeta_2+\zeta_2^2\right)\right)\right)\right)/\\
&\left(9\Delta_2^6\Delta_3\zeta_2\left(6\Delta_2^3-6\Delta_3^3+\Delta_2\Delta_3\left(-2\zeta_1+\zeta_2\right)\right)\right);
\end{aligned}
$$

Instructed by the case of zeroth level we try to inspect the solution which at 0-th level reduces to

$$
\left\{\Omega_1 \to 1\,,\,\Omega_2 \to \sqrt{\frac{\Delta_3}{\Delta_2}}\right\} \iff \left\{\Upsilon_1 \to \frac{\Delta_2}{\Delta_3}\,,\,\Upsilon_2 \to \frac{\Delta_3^2}{\Delta_2^2}\right\} \tag{D.49}
$$

To this effect we consider a power series expansion of the solution for small moment maps, around the 0th level solution:

$$
\Upsilon_1 \to \frac{\Delta_2}{\Delta_3}+Y_1\varepsilon,\, \Upsilon_2 \to \frac{\Delta_3^2}{\Delta_2^2}+Y_2\varepsilon,\, \zeta_1 \to k_1\varepsilon,\, \zeta_2 \to k_2\varepsilon \tag{D.50}
$$

where ε is an infinitesimal parameter. Inserting eq. (D.50) into the moment map equation we obtain the approximate solution:

$$
\Omega_1 \to 1-\frac{k_1\varepsilon}{18\left(\Delta_2+\Delta_3\right)}\,, \quad \Omega_2 \to \frac{18\Delta_3\left(\Delta_2+\Delta_3\right)-\Delta_3k_2\varepsilon}{18\sqrt{\Delta_2\Delta_3}\left(\Delta_2+\Delta_3\right)} \tag{D.51}
$$

which in terms of the complex quiver group transformations can also be interpreted as follows:

$$
\mu_1 \to 1-\varepsilon\,\frac{k_1}{18\left(\Delta_2+\Delta_3\right)}\,, \quad \mu_2 \to \frac{u_3}{u_2}\left(1-\varepsilon\,\frac{k_2}{18\left(\Delta_2+\Delta_3\right)}\right) \tag{D.52}
$$

In some way, to be better clarified, the above deformation should describe the inflation of the two homology cycles predicted by the general theorem.

D.1.7 The formal solution for the Kähler potential

In any case, assuming the scalar factors $\Omega_{1,2} = \Omega_{1,2}\left(\boldsymbol{u},\overline{\boldsymbol{u}},\boldsymbol{\zeta}\right)$ known in terms of the coordinates and of the moment map parameters, we can calculate the final Kähler potential. Substituting (D.35) into eq. (D.28) we obtain the restriction to the level surface $\mathscr{N}$ of the original Kähler potential:

$$\mathscr{K}|_{\mathscr{N}} = \frac{3u_3^2\Omega_1^4\bar{u}_3^2}{u_2\Omega_2^2\bar{u}_2} + \frac{3u_3^2\Omega_1^2\bar{u}_3^2}{u_2\Omega_2^4\bar{u}_2} + \frac{3u_2^2\Omega_2^2\Omega_1^2\bar{u}_2^2}{u_3\bar{u}_3} + \frac{3u_3^2\bar{u}_3^2}{u_2\Omega_2^2\Omega_1^2\bar{u}_2} + \frac{3u_2^2\Omega_2^4\bar{u}_2^2}{u_3\Omega_1^2\bar{u}_3} + \frac{3u_2^2\Omega_2^2\bar{u}_2^2}{u_3\Omega_1^4\bar{u}_3} + 9u_1\bar{u}_1 \tag{D.53}$$

Then the final Kähler potential of the resolved smooth manifold is:

$$\mathscr{K}_{\mathscr{M}} = \mathscr{K}|_{\mathscr{N}} + \zeta^1 \log\Omega_1 + \zeta^2 \log\Omega_2 \tag{D.54}$$

Note that when $\zeta_{1,2} = 0$ we have (see eq. (D.49)):

$$\Omega_1 = 1 \quad , \quad \Omega_2 = \sqrt{\frac{u_3\bar{u}_3}{u_2\bar{u}_2}} \tag{D.55}$$

which inserted into eq. (D.53) yields:

$$\lim_{\boldsymbol{\zeta}\to\mathbf{0}} \mathscr{K}|_{\mathscr{M}} = \lim_{\boldsymbol{\zeta}\to\mathbf{0}} \mathscr{K}|_{\mathscr{N}} = 9u_1\bar{u}_1 + 9u_2\bar{u}_2 + 9u_3\bar{u}_3 \tag{D.56}$$

namely we obtain the Kähler potential of the flat $\mathbb{C}^3/\mathbb{Z}_3$ orbifold of which $u_{1,2,3}$ are the $\mathbb{Z}_3$ invariant coordinates.

D.1.8 Comparison with GH metrics

Although for the case under study, the explicit form of the Kähler potential is not available in terms of radicals, since it involves the roots of a sextic equation, yet the metric can be easily written in terms of real coordinates by utilizing the Gibbons-Hawking form of ALE metrics. Let us recall eq. (7.40) which predicts the number of parameters in the hyperKähler quotient resolution of a $\mathbb{C}^2/\mathbb{Z}_p$ singularity. For $p = 3$ this number is 4, yet in our case the number of parameters is less since we take only the Kähler quotient and we keep fixed the analogue of the holomorphic moment map equations, namely the constraint $[A,B] = [B,C] = [C,A]$. Hence although, as we have observed, the resolution of the singularity $\mathbb{C}^3/\mathbb{Z}_3$ reduces to a direct product $\mathbb{C} \times \left(Y \to \mathbb{C}^2/\mathbb{Z}_3\right)$ yielding $\mathbb{C} \times$ $ALE_{\mathbb{Z}_3}$ (indeed, once diagonalized the action of $\mathbb{Z}_3$ is effective only on two complex coordinates), the $ALE_{\mathbb{Z}_3}$ that we obtain in this process is a particular one depending only on 2 of the 4 available parameters. From the holomorphic point of view the missing parameters are clearly localized. They are those of a holomorphic moment map level which is not switched on. The other can always be suppressed by a coordinated change. This is the reason why in this type of resolutions, the algebraic equation is not touched (it is preserved identical to the case of the orbifold, by utilizing only the locus $\mathscr{D}^0_\Gamma$ which is the orbit of the locus L^Γ under the action of the quiver group $\mathscr{G}_\Gamma$). In the case of Eguchi-Hanson, as we have seen, there is no loss of generality, but for all the other cases we have a reduced number of moduli with respect to the general ALE. This implies that also the GH-metric which is equivalent to these specialized ALE manifolds should be in some sense a special one and should depend only on (p-1) parameters. We conjecture that the GH multicenter spaces equivalent to the special ALE of the above discussion are those where the centers are all aligned on the same line, say along the z-axis. Let us follow this idea.

D.1.9 The harmonic potential and the connection one-form

As for the harmonic potential $\mathscr{V}_{\mathbb{Z}_3}$ we adopt the following one:

$$\mathscr{V}_{\mathbb{Z}_3} = \frac{1}{\sqrt{x^2+y^2+(z-\delta_2)^2}} + \frac{1}{\sqrt{x^2+y^2+(z-\delta_3)^2}} + \frac{1}{\sqrt{x^2+y^2+(z-\delta_1)^2}} \tag{D.57}$$

implying that the three centers of the metric are at $\mathbf{c}_i = \{0,0,\delta_i\}$. The corresponding connection one-form is as follows:

$$\omega_{\mathbb{Z}_3} = \frac{(z-\delta_1)(x\mathrm{d}y - y\mathrm{d}x)}{(x^2+y^2)\sqrt{x^2+y^2+(z-\delta_1)^2}} + \frac{(z-\delta_2)(x\mathrm{d}y - y\mathrm{d}x)}{(x^2+y^2)\sqrt{x^2+y^2+(z-\delta_2)^2}} + \frac{(z-\delta_3)(x\mathrm{d}y - y\mathrm{d}x)}{(x^2+y^2)\sqrt{x^2+y^2+(z-\delta_3)^2}} \quad \text{(D.58)}$$

This information suffices to write the explicit form of the metric as given in eq. (C.1) and of the the Kähler 2-form as given in eq. (C.6). Inspired by the example of the Eguchi-Hanson case, we can now conjecture the location of the homology two-cycles within the GH-manifold. We embed two $\mathbb{S}^2$ parameterized by the angles θ,ϕ by setting:

$$\begin{array}{rcl} D_E^1 & = & \left\{x=y=0 \quad , \quad z=\frac{1}{2}(\delta_1+\delta_2)+\frac{1}{2}(\delta_1-\delta_2)\cos\theta \quad , \quad \tau=\phi\right\} \\ D_E^2 & = & \left\{x=y=0 \quad , \quad z=\frac{1}{2}(\delta_2+\delta_3)+\frac{1}{2}(\delta_2-\delta_3)\cos\theta \quad , \quad \tau=\phi\right\} \end{array} \quad \text{(D.59)}$$

D.2 Construction à la Kronheimer of the crepant resolution $\mathscr{M}_{\mathbb{Z}_7} \to \frac{\mathbb{C}^3}{\mathbb{Z}_7}$

Next we address the case of the singularity $\frac{\mathbb{C}^3}{\mathbb{Z}_7}$, provided by the embedding $\mathbb{Z}_7 \hookrightarrow \mathrm{SU}(3)$ encoded in the form of the generator Y of eq. (A.7). It is still an abelian case as the previous one, yet the resolution $\mathscr{M}_{\mathbb{Z}_7} \to \frac{\mathbb{C}^3}{\mathbb{Z}_3}$ no longer factorizes:

$$\mathscr{M}_{\mathbb{Z}_7} \neq \mathbb{C}\times ALE_{\mathbb{Z}_7} \quad \text{(D.60)}$$

although $ALE_{\mathbb{Z}_7}$ does independently exist. The reason is that in the quotient $\frac{\mathbb{C}^3}{\mathbb{Z}_7}$ there is no linear subspace of $\mathbb{C}^3$ which is left invariant. Hence this is a truly new case which displays intrinsically three-dimensional features. The most dramatic and relevant of them is the prediction from theorem 4.1 and from eq. (A.21) of the existence of 3-harmonic (2,2)-forms, along side with three harmonic (1,1)-forms:

$$h^{1,1}(\mathscr{M}_{\mathbb{Z}_7}) = 3 \quad ; \quad h^{2,2}(\mathscr{M}_{\mathbb{Z}_7}) = 3 \quad \text{(D.61)}$$

Notwithstanding these novelties, the construction of the resolution à la Kronheimer, which is the one directly mirrored in the structure of the $D=3$ Chern-Simons gauge theory supposedly dual to one M2-brane that probes the corresponding singularity, goes along the same lines as before.

Following the general strategy, the first step in the construction consists of deriving the invariant space $\mathscr{S}_\Gamma = \mathrm{Hom}_\Gamma(R,\mathscr{Q}\otimes R)$ made of those triples $\{A,B,C\}$ of 7×7 matrices that satisfy eq. (6.6), namely:

$$\mathrm{Y}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} R(\mathrm{Y}^{-1})AR(\mathrm{Y}) \\ R(\mathrm{Y}^{-1})BR(\mathrm{Y}) \\ R(\mathrm{Y}^{-1})CR(\mathrm{Y}) \end{pmatrix} \quad \text{(D.62)}$$

where Y is the $\mathbb{Z}_7$-generator displayed in eq (A.7) and $R(\mathrm{Y})$ is its representation in the natural basis of the regular representation:

$$R(\mathrm{Y}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{D.63}$$

The constraint (D.62) reduces the number of parameters from 147 to 21. If we perform the diagonalization of the regular representation $R_{diag}(\mathrm{Y}) \equiv \boldsymbol{\chi}^\dagger R(\mathrm{Y})\boldsymbol{\chi}$ by means of the normalize character table:

$$\boldsymbol{\chi} = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} \\ 1 & e^{\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} \\ 1 & e^{\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} \\ 1 & e^{-\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} \\ 1 & e^{-\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} \\ 1 & e^{-\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} \end{pmatrix} \tag{D.64}$$

the result of the constraint (D.62) with $R_{diag}(\mathrm{Y})$ in place of $R(\mathrm{Y})$ can be expressed in the following way:

$$\begin{aligned} (A,B,C) & \in \mathrm{Hom}_\Gamma(R, \mathscr{Q}\otimes R) \\ & \Downarrow \end{aligned}$$

$$A = \begin{pmatrix} 0 & 0 & m_{1,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{2,4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{3,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{4,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{5,7} \\ m_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{7,2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.65}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & n_{1,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_{2,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_{3,7} \\ n_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{5,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_{6,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{7,4} & 0 & 0 & 0 \end{pmatrix} \tag{D.66}$$

$$C = \begin{pmatrix} 0 & r_{1,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{2,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{3,4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{4,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{5,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{6,7} \\ r_{7,1} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.67}$$

in terms of a new set of 21 parameters evident from the above formulae.

D.2.1 Derivation of the locus $\mathscr{D}_{\mathbb{Z}_7}$ and of the variety $\mathbb{V}_{7+2}$

Next we define the reduction of the invariant space $\mathrm{Hom}_\Gamma(R, \mathscr{Q} \otimes R)$ to the locus cut out by the holomorphic constraint (6.12) which was named $\mathscr{D}_\Gamma$:

$$\mathscr{D}_{\mathbb{Z}_7} \equiv \left\{ p = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathrm{Hom}_{\mathbb{Z}_7}(R, \mathscr{Q} \otimes R) \;/\; [A,B] = [B,C] = [C,A] = 0 \right\} \tag{D.68}$$

Differently from the more complicated cases of maximal subgroups of L_{168} and analogously to the previous $\mathbb{Z}_3$ case, here we can explicitly solve the quadratic equations provided by the commutator constraints and we discover that there is a unique principal branch of the solution, named $\mathscr{D}^0_\Gamma$, that has the maximal dimension $9 = |\Gamma| + 2$. However, in addition to that there are 785 more branches of the solution with smaller dimension. Also here, $\mathscr{D}^0_\Gamma$ is the orbit with respect to the group $\mathscr{G}_\Gamma$ of the subspace L_Γ defined in eq. (5.34). Hence the variety $\mathbb{V}_{|\Gamma|+2}$ of which we are supposed to take the Kähler quotient is not defined by eq. (D.68), rather by the principal branch of that variety.

D.2.2 Derivation of the quiver group $\mathscr{G}_{\mathbb{Z}_7}$

In view of what we stated above our next point is precisely the derivation of the group $\mathscr{G}_\Gamma$ defined in eq. (6.19), namely:

$$\mathscr{G}_{\mathbb{Z}_7} = \{ g \in \mathrm{SL}(7,\mathbb{C}) \quad | \quad \forall \gamma \in \mathbb{Z}_3 : \; [D_R(\gamma), D_{\mathrm{def}}(g)] = 0 \} \tag{D.69}$$

Let us proceed to this construction. In the diagonal basis of the regular representation this is a very easy task, since the group is simply given by the diagonal 7×7 matrices with determinant one. A convenient parametrization of such a group is the following one:

$$g = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_2}{\mu_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu_3}{\mu_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu_4}{\mu_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu_5}{\mu_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mu_6}{\mu_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_6} \end{pmatrix} \in \mathscr{G}_{\mathbb{Z}_3} \quad ; \quad \mu_{1,\dots,6} \in \mathbb{C} \tag{D.70}$$

The explicit form (D.70) allows to construct the $\mathrm{Orbit}_{\mathscr{G}_{\mathbb{Z}_7}}(L_{\mathbb{Z}_7})$ of the locus

$$L_{\mathbb{Z}_7} \equiv \{A_0, B_0, C_0\} \tag{D.71}$$

made by those triples of matrices belonging to $\mathrm{Hom}_{\mathbb{Z}_7}(\mathscr{Q} \otimes R, R)$ that are diagonal in the natural basis of the regular representation. Considering eqs. (D.65-D.67) such a locus, which has complex dimension 3, is obtained, by setting:

$$\begin{aligned} m_{1,3} = m_{2,4} = m_{3,5} = m_{4,6} = m_{5,7} = m_{6,1} = m_{7,2} &= u_1 \\ n_{1,5} = n_{2,6} = n_{3,7} = n_{4,1} = n_{5,2} = n_{6,3} = n_{7,4} &= u_2 \\ r_{1,2} = r_{2,3} = r_{3,4} = r_{4,5} = r_{5,6} = r_{6,7} = r_{7,1} &= u_3 \end{aligned} \tag{D.72}$$

where $u_{1,2,3}$ are three complex parameter that will play the role of coordinates of the resolved manifold. The six complex parameters of the quiver group plus these three make the total of nine parameters of $\mathbb{V}_9 \equiv \mathrm{Orbit}_{\mathscr{G}_{\mathbb{Z}_7}}(L_{\mathbb{Z}_7})$, which, as we have explicitly verified is the same as the principal branch $\mathscr{D}^0_{\mathbb{Z}_7}$ of the quadratic locus $[A,B] = [B,C] = [C,A] = 0$.

In conclusion the variety $\mathbb{V}_9$ of which we are supposed to perform the Kähler quotient is spanned by the following triple of matrices, depending on the 9 complex parameters $u_{1,2,3}, \mu_{1,\dots,6}$:

$$\mathbb{V}_9 = \{A, B, C\}$$

$$A = \begin{pmatrix} 0 & 0 & \frac{u_1\mu_3}{\mu_1\mu_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{u_1\mu_1\mu_4}{\mu_2\mu_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{u_1\mu_2\mu_5}{\mu_3\mu_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{u_1\mu_3\mu_6}{\mu_4\mu_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_1\mu_4}{\mu_5\mu_6} \\ \frac{u_1\mu_1\mu_5}{\mu_6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{u_1\mu_2\mu_6}{\mu_1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.73}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{u_2\mu_5}{\mu_1\mu_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{u_2\mu_1\mu_6}{\mu_2\mu_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_2\mu_2}{\mu_3\mu_6} \\ \frac{u_2\mu_1\mu_3}{\mu_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{u_2\mu_2\mu_4}{\mu_1\mu_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{u_2\mu_3\mu_5}{\mu_2\mu_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{u_2\mu_4\mu_6}{\mu_3} & 0 & 0 & 0 \end{pmatrix} \tag{D.74}$$

$$C \quad = \quad \begin{pmatrix} 0 & \frac{u_3\mu_2}{\mu_1^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{u_3\mu_1\mu_3}{\mu_2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{u_3\mu_2\mu_4}{\mu_3^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{u_3\mu_3\mu_5}{\mu_4^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{u_3\mu_4\mu_6}{\mu_5^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_3\mu_5}{\mu_6^2} \\ u_3\mu_1\mu_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.75}$$

D.2.3 The algebraic equation of the orbifold locus

Let us now consider the action of the $\mathbb{Z}_3$ group on $\mathbb{C}^3$ as defined by the generator X in eq. (A.7). If $\{z_1, z_2, z_3\}$ are the coordinates of a point in $\mathbb{C}^3$, we see that there are four invariant polynomials:

$$\begin{aligned} J_1 &= z_1^7 \\ J_2 &= z_2^7 \\ J_3 &= z_3^7 \\ J_4 &= z_1\, z_2\, z_3 \end{aligned} \tag{D.76}$$

which satisfy the following equation:

$$J_1\, J_2\, J_3 - J_4^7 = 0 \tag{D.77}$$

This equation can be regarded as the cubic equation which cuts out in $\mathbb{C}^4$ the locus corresponding to the singular orbifold $\mathbb{C}^3/\mathbb{Z}_7$.

D.2.4 Map of the variety $\mathbb{V}_9$ into the algebraic locus corresponding to the orbifold

Having established the above relations we verify that, in a completely analogous way to what happens in the case of the standard Kronheimer construction, we can reproduce the defining equation (D.77) in terms of invariants of the three matrices (D.73-D.75) spanning the $\mathbb{V}_9$ variety. It suffices to identify:

$$\begin{aligned} J_1 &= \mathrm{Det}[A] = u_1^7 \\ J_2 &= \mathrm{Det}[B] = u_2^7 \\ J_3 &= \mathrm{Det}[C] = u_3^7 \\ J_4 &= \mathrm{Det}[A\,B\,C] = u_1\, u_2\, u_3 \end{aligned} \tag{D.78}$$

Eq. (D.78) describes an explicit map of the variety $\mathbb{V}_9$ into the algebraic locus $\mathbb{C}^3/\mathbb{Z}_7$:

$$\mathbb{V}_9 \rightarrow \frac{\mathbb{C}^3}{\mathbb{Z}_7} \tag{D.79}$$

D.2.5 The Kähler quotient

The next step consists of performing the Kähler quotient of the Kähler manifold $\mathbb{V}_9$ with respect to the compact subgroup of the quiver group $\mathscr{G}_{\mathbb{Z}_7}$, which, as we several times emphasized, is the *gauge group* of the corresponding three-dimensional Chern-Simons gauge theory:

$$\mathscr{F}_{\mathbb{Z}_7} \equiv \mathscr{G}_{\mathbb{Z}_7} \bigcap \mathrm{SU}(7) \tag{D.80}$$

A generic element $g \in \mathscr{F}_{\mathbb{Z}_7}$, is of the form (D.70) with:

$$\mu_i = \exp[i\,\theta_i] \tag{D.81}$$

The Kähler structure of $\mathbb{V}_9$ is provided by the pullback on the $\mathbb{V}_9$ surface of the Kähler potential of the entire flat Kähler manifold $\mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q} \otimes R)$, namely we have:

$$\begin{aligned}
\mathscr{K}_{\mathbb{V}_9} &\equiv \mathrm{Tr}\,\left(\left[A^\dagger, A\right] + \left[B^\dagger, B\right] + \left[C^\dagger, C\right]\right) \\
&= \frac{\mu_2\mu_4u_3\overline{\mu}_4\overline{\mu}_2\overline{u}_3}{\mu_3^2\overline{\mu}_3^2} + \frac{\mu_2\mu_5u_1\overline{\mu}_5\overline{\mu}_2\overline{u}_1}{\mu_3\mu_4\overline{\mu}_3\overline{\mu}_4} + \frac{\mu_2\mu_6u_1\overline{\mu}_6\overline{\mu}_2\overline{u}_1}{\mu_1\overline{\mu}_1} + \frac{\mu_2\mu_4u_2\overline{\mu}_4\overline{\mu}_2\overline{u}_2}{\mu_1\mu_5\overline{\mu}_1\overline{\mu}_5} \\
&\quad + \frac{\mu_2u_2\overline{\mu}_2\overline{u}_2}{\mu_3\mu_6\overline{\mu}_3\overline{\mu}_6} + \frac{\mu_2u_3\overline{\mu}_2\overline{u}_3}{\mu_1^2\overline{\mu}_1^2} + \frac{\mu_5u_2\overline{\mu}_5\overline{u}_2}{\mu_1\mu_4\overline{\mu}_1\overline{\mu}_4} \\
&\quad + \frac{\mu_3\mu_5u_3\overline{\mu}_3\overline{\mu}_5\overline{u}_3}{\mu_4^2\overline{\mu}_4^2} + \mu_1\mu_6u_3\overline{\mu}_1\overline{\mu}_6\overline{u}_3 + \frac{\mu_4\mu_6u_2\overline{\mu}_4\overline{\mu}_6\overline{u}_2}{\mu_3\overline{\mu}_3} + \frac{\mu_3\mu_6u_1\overline{\mu}_3\overline{\mu}_6\overline{u}_1}{\mu_4\mu_5\overline{\mu}_4\overline{\mu}_5} \\
&\quad + \frac{\mu_4\mu_6u_3\overline{\mu}_4\overline{\mu}_6\overline{u}_3}{\mu_5^2\overline{\mu}_5^2} + \frac{\mu_1\mu_3u_2\overline{\mu}_1\overline{\mu}_3\overline{u}_2}{\mu_4\overline{\mu}_4} \\
&\quad + \frac{\mu_1\mu_5u_1\overline{\mu}_1\overline{\mu}_5\overline{u}_1}{\mu_6\overline{\mu}_6} + \frac{\mu_4u_1\overline{\mu}_4\overline{u}_1}{\mu_5\mu_6\overline{\mu}_5\overline{\mu}_6} + \frac{\mu_5u_3\overline{\mu}_5\overline{u}_3}{\mu_6^2\overline{\mu}_6^2} + \frac{\mu_3u_1\overline{\mu}_3\overline{u}_1}{\mu_1\mu_2\overline{\mu}_1\overline{\mu}_2} \\
&\quad + \frac{\mu_1\mu_4u_1\overline{\mu}_1\overline{\mu}_4\overline{u}_1}{\mu_2\mu_3\overline{\mu}_3\overline{\mu}_2} + \frac{\mu_1\mu_6u_2\overline{\mu}_1\overline{\mu}_6\overline{u}_2}{\mu_2\mu_5\overline{\mu}_5\overline{\mu}_2} + \frac{\mu_3\mu_5u_2\overline{\mu}_3\overline{\mu}_5\overline{u}_2}{\mu_2\mu_6\overline{\mu}_6\overline{\mu}_2} + \frac{\mu_1\mu_3u_3\overline{\mu}_1\overline{\mu}_3\overline{u}_3}{\mu_2^2\overline{\mu}_2^2}
\end{aligned} \tag{D.82}$$

$\mathscr{K}_{\mathbb{V}_9}$ is obviously invariant under the unitary transformations of the *gauge group* :

$$\forall g \in \mathscr{F}_{\mathbb{Z}_7}: \quad \{A, B, C\} \to \left\{g^\dagger A g, g^\dagger B g, g^\dagger C g\right\} \tag{D.83}$$

which, for that reason, is an *isometry group* of the corresponding Kähler metric on $\mathrm{Hom}_{\mathbb{Z}_7}(R, \mathscr{Q} \otimes R)$ and of its restriction to $\mathbb{V}_9$. The last point follows from the fact that, by construction, $\mathscr{F}_{\mathbb{Z}_7}$ maps $\mathbb{V}_9 \subset \mathrm{Hom}_{\mathbb{Z}_3}(R, \mathscr{Q} \otimes R)$ into itself.

A basis of six linearly independent generators of the Lie algebra $\mathbb{F}_{\mathbb{Z}_3}$ is provided by the following six matrices:

$$\begin{aligned}
\mathfrak{f}_1 &= \mathrm{diag}\,(i, -i, 0, 0, 0, 0, 0) \\
\mathfrak{f}_2 &= \mathrm{diag}\,(0, i, -i, 0, 0, 0, 0) \\
\mathfrak{f}_3 &= \mathrm{diag}\,(0, 0, i, -i, 0, 0, 0) \\
\mathfrak{f}_4 &= \mathrm{diag}\,(0, 0, 0, i, -i, 0, 0) \\
\mathfrak{f}_5 &= \mathrm{diag}\,(0, 0, 0, 0, i, -i, 0) \\
\mathfrak{f}_6 &= \mathrm{diag}\,(0, 0, 0, 0, 0, i, -i)
\end{aligned} \tag{D.84}$$

and the moment maps corresponding to the isometries generated by them are defined as follows:

$$\mathfrak{P}_A = -i\,\mathrm{Tr}\left(\mathfrak{f}_A\left(\left[A^\dagger, A\right] + \left[B^\dagger, B\right] + \left[C^\dagger, C\right]\right)\right), \quad (A = 1, \ldots, 6) \tag{D.85}$$

We do not write the explicit form of $\mathfrak{P}_A$ since it is rather involved. First we change basis for the generators of the Lie algebra $\mathbb{F}_{\mathbb{Z}_7}$ observing that they are not orthogonal with respect to the Killing form defined by the trace.

Indeed we have:

$$\kappa_{AB} \equiv \mathrm{Tr}\left(\mathfrak{f}_A^\dagger\,\mathfrak{f}_B\right) = \left(\begin{array}{cccccc} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right) \tag{D.86}$$

which is nothing else but the Cartan matrix of the A_6 Lie algebra. Consequently we define a new basis of moment maps, dual to the above one:

$$\mathfrak{P}^A = \left(\kappa^{-1}\right)^{AB} \mathfrak{P}_B \tag{D.87}$$

and we express them in terms of the following variables:

$$\Delta_{1,2,3} \equiv |u_{1,2,3}|^2 \quad ; \quad \mu_i = \overline{\mu}_i = \Omega_i \in \mathbb{R}_+ \quad (i = 1,\dots 6) \tag{D.88}$$

The choice (D.88) streams from the following three facts that we find it convenient to recall once again:

A) When $\mu_i = 1$ $(i = 1,\dots,6)$, namely when the quiver group element with which we have rotated the locus $L_{\mathbb{Z}_7}$ is the identity, the moment maps $\mathfrak{P}^A$ are all zero.

B) The moment maps $\mathfrak{P}^A$ are invariant under the action of the gauge group $\mathscr{F}_{\mathbb{Z}_7} \subset \mathscr{G}_{\mathbb{Z}_7}$.

C) It follows from A) and B) that the only transformations of the quiver group that lift the levels of the moment maps from zero are those in the coset $\frac{\mathscr{G}_{\mathbb{Z}_7}}{\mathscr{F}_{\mathbb{Z}_7}}$.

In this way we obtain the following result:

$$\mathfrak{P}^1 = \frac{\Omega_1^2\left(\frac{\Delta_2\left(\Omega_5^2-\Omega_1^4\Omega_3^2\right)}{\Omega_4^2}+\Delta_1\left(\frac{\Omega_3^2}{\Omega_2^2}-\frac{\Omega_1^4\Omega_5^2}{\Omega_6^2}\right)\right)+\Delta_3\left(\Omega_2^2-\Omega_1^6\Omega_6^2\right)}{\Omega_1^4} \tag{D.89}$$

$$\mathfrak{P}^2 = \frac{\frac{\Delta_3\left(\Omega_3^2-\Omega_2^4\Omega_6^2\right)\Omega_1^4}{\Omega_2^4}+\Delta_1\left(-\frac{\Omega_5^2\Omega_1^4}{\Omega_6^2}-\Omega_2^2\Omega_6^2+\frac{\Omega_4^2\Omega_1^4+\Omega_3^4}{\Omega_2^2\Omega_3^2}\right)+\Delta_2\left(\left(\frac{\Omega_6^2}{\Omega_2^2\Omega_5^2}-\frac{\Omega_3^2}{\Omega_4^2}\right)\Omega_1^4+\frac{\Omega_5^4-\Omega_2^2\Omega_4^4}{\Omega_4^2\Omega_5^2}\right)}{\Omega_1^2} \tag{D.90}$$

$$\begin{aligned} \mathfrak{P}^3 = {} & \Delta_3\left(\frac{\Omega_2^2\Omega_4^2}{\Omega_3^4}-\Omega_1^2\Omega_6^2\right)+\Delta_1\left(\left(\frac{\Omega_4^2}{\Omega_2^2\Omega_3^2}-\frac{\Omega_5^2}{\Omega_6^2}\right)\Omega_1^2+\frac{\Omega_2^2\Omega_5^2}{\Omega_3^2\Omega_4^2}-\frac{\Omega_2^2\Omega_6^2}{\Omega_1^2}\right) \\ & +\Delta_2\left(\left(\frac{\Omega_6^2}{\Omega_2^2\Omega_5^2}-\frac{\Omega_3^2}{\Omega_4^2}\right)\Omega_1^2+\frac{\Omega_2^4-\Omega_3^4\Omega_5^2}{\Omega_2^2\Omega_3^2\Omega_6^2}+\frac{\Omega_5^4-\Omega_2^2\Omega_4^4}{\Omega_4^2\Omega_5^2\Omega_1^2}\right) \end{aligned} \tag{D.91}$$

$$\begin{aligned} \mathfrak{P}^4 = {} & \Delta_3\left(\frac{\Omega_3^2\Omega_5^2}{\Omega_4^4}-\Omega_1^2\Omega_6^2\right)+\Delta_1\left(\left(\frac{\Omega_5^2}{\Omega_3^2\Omega_4^2}-\frac{\Omega_6^2}{\Omega_1^2}\right)\Omega_2^2+\frac{\Omega_3^2\Omega_6^2}{\Omega_4^2\Omega_5^2}-\frac{\Omega_1^2\Omega_5^2}{\Omega_6^2}\right) \\ & +\Delta_2\left(\frac{\Omega_1^2\Omega_6^2}{\Omega_2^2\Omega_5^2}+\frac{\Omega_5^4-\Omega_2^2\Omega_4^4}{\Omega_1^2\Omega_4^2\Omega_5^2}+\frac{\Omega_2^4-\Omega_4^2\Omega_6^4\Omega_2^2-\Omega_3^4\Omega_5^2}{\Omega_2^2\Omega_3^2\Omega_6^2}\right) \end{aligned} \tag{D.92}$$

$$\mathfrak{P}^5 = \frac{\frac{\Delta_3\left(\Omega_4^2-\Omega_1^2\Omega_5^4\right)\Omega_6^4}{\Omega_5^4}+\Delta_1\left(-\frac{\Omega_2^2\Omega_6^4}{\Omega_1^2}+\frac{\Omega_3^2\Omega_6^4}{\Omega_4^2\Omega_5^2}-\Omega_1^2\Omega_5^2+\frac{\Omega_4^2}{\Omega_5^2}\right)+\Delta_2\left(-\frac{\Omega_4^2\Omega_6^4}{\Omega_3^2}+\frac{\Omega_1^2\Omega_6^4-\Omega_3^2\Omega_5^4}{\Omega_2^2\Omega_5^2}+\frac{\Omega_2^2}{\Omega_3^2}\right)}{\Omega_6^2} \tag{D.93}$$

$$\mathfrak{P}^6 = \frac{\Omega_6^2\left(\Delta_1\left(\frac{\Omega_4^2}{\Omega_5^2}-\frac{\Omega_2^2\Omega_6^4}{\Omega_1^2}\right)+\frac{\Delta_2\left(\Omega_2^2-\Omega_4^2\Omega_6^4\right)}{\Omega_3^2}\right)+\Delta_3\left(\Omega_5^2-\Omega_1^2\Omega_6^6\right)}{\Omega_6^4} \tag{D.94}$$

and we are led to the following system of six higher order algebraic equations:

$$\mathfrak{P}^A(\Delta,\Omega) = \zeta^A \tag{D.95}$$

that has to be solved for Ω_A in terms of $\Delta_{1,2,3}$ and of the level parameters ζ^A. At vanishing level $\boldsymbol{\zeta} = \mathbf{0}$ we know that the solution is $\boldsymbol{\Omega} = \mathbf{1}$, hence the best we can do, since we deal with higher order equations that admit no solution by radicals, is to attempt a power series solution in terms of the levels $\boldsymbol{\zeta}$. Formally, however, the problem of the Kähler quotient is solved. Assuming the scalar factors $\Omega_{1,\dots,6} = \Omega_{1,\dots,6}\left(|\boldsymbol{u}|^2,\boldsymbol{\zeta}\right)$ to be known in terms of the coordinates and of the moment map parameters, we can calculate the final Kähler potential. Substituting (D.88) into eq. (D.82) we obtain the restriction to the level surface $\mathscr{N}$ of the original Kähler potential:

$$\begin{aligned}\mathscr{K}|_{\mathscr{N}} \;=\;& \frac{\Delta_3\Omega_4^2\Omega_2^2}{\Omega_3^4}+\frac{\Delta_1\Omega_5^2\Omega_2^2}{\Omega_3^2\Omega_4^2}+\frac{\Delta_1\Omega_6^2\Omega_2^2}{\Omega_1^2}+\frac{\Delta_2\Omega_4^2\Omega_2^2}{\Omega_1^2\Omega_5^2}+\frac{\Delta_2\Omega_2^2}{\Omega_3^2\Omega_6^2}+\frac{\Delta_3\Omega_2^2}{\Omega_1^4}+\frac{\Delta_2\Omega_5^2}{\Omega_1^2\Omega_4^2}+\frac{\Delta_3\Omega_3^2\Omega_5^2}{\Omega_4^4}\\ &+\Delta_3\Omega_1^2\Omega_6^2+\frac{\Delta_2\Omega_4^2\Omega_6^2}{\Omega_3^2}+\frac{\Delta_1\Omega_3^2\Omega_6^2}{\Omega_4^2\Omega_5^2}+\frac{\Delta_3\Omega_4^2\Omega_6^2}{\Omega_5^4}+\frac{\Delta_2\Omega_1^2\Omega_3^2}{\Omega_4^2}+\frac{\Delta_1\Omega_1^2\Omega_5^2}{\Omega_6^2}+\frac{\Delta_1\Omega_4^2}{\Omega_5^2\Omega_6^2}\\ &+\frac{\Delta_3\Omega_5^2}{\Omega_6^4}+\frac{\Delta_1\Omega_3^2}{\Omega_1^2\Omega_2^2}+\frac{\Delta_1\Omega_1^2\Omega_4^2}{\Omega_3^2\Omega_2^2}+\frac{\Delta_2\Omega_1^2\Omega_6^2}{\Omega_5^2\Omega_2^2}+\frac{\Delta_2\Omega_3^2\Omega_5^2}{\Omega_6^2\Omega_2^2}+\frac{\Delta_3\Omega_1^2\Omega_3^2}{\Omega_2^4}\end{aligned} \tag{D.96}$$

Then the final Kähler potential of the resolved smooth manifold is:

$$\mathscr{K}_{\mathscr{M}} = \mathscr{K}|_{\mathscr{N}} + \sum_{A=1}^{6} \zeta^A \log\Omega_A \tag{D.97}$$

When $\zeta^A = 0$ we have, as we already said, $\Omega_A = 1$ ($A = 1,\dots,6$ which, inserted into eq. (D.96), yields:

$$\lim_{\boldsymbol{\zeta}\to\mathbf{0}} \mathscr{K}|_{\mathscr{M}} = \lim_{\boldsymbol{\zeta}\to\mathbf{0}} \mathscr{K}|_{\mathscr{N}} = 7u_1\bar{u}_1+7u_2\bar{u}_2+7u_3\bar{u}_3 \tag{D.98}$$

namely we obtain the Kähler potential of the flat $\mathbb{C}^3/\mathbb{Z}_7$ orbifold of which $u_{1,2,3}$ are the $\mathbb{Z}_7$ invariant coordinates.

E A non-abelian example: crepant resolution à la Kronheimer of $\mathscr{M}_{\mathrm{Dih}_3}$

In the ADE classification of SU(2) subgroups we do not find the dihedral groupsDih$_m$ whose order is $|\mathrm{Dih}_m|$ $= 2m$. We rather find their binary extensions $\mathrm{Dih}_m{}^b$ whose order is $|\mathrm{Dih}_m{}^b|$ = 4m.

The $\mathrm{Dih}_m{}^b$ groups correspond to the Lie algebras D_{m+2} in the McKay correspondence. The smallest non-abelian case corresponds to $\mathrm{Dih}_3{}^b$ with 12 elements, since $\mathrm{Dih}_2{}^b$ with eight elements is abelian. The matrices occurring in the generalized Kronheimer construction are already rather big and the degree of the resolving algebraic equations is expected to be rather high.

On the other hand we can easily embed the groups Dih_m into SU(3). Hence we address immediately the case of the dihedral groups in $\frac{\mathbb{C}^3}{\mathrm{Dih}_m}$ which allows to consider the case $m = 3$ with 6 elements. As it is well known Dih_3 is isomorphic to S_3, the symmetric group on three elements and this is the smallest nonabelian group. It remains to understand what is the generalized Weyl group corresponding to the quiver matrix generated by this case which, as we show below is the following one:

$$\mathrm{CQ} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{E.1}$$

The corresponding extended diagram is the following one:

$$\mathrm{CE} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix} \tag{E.2}$$

E.1 Definition of $\mathrm{Dih}_m \subset \mathrm{SU}(3)$

The presentation of the dihedral group Dih_m is the following one

$$A^m = 1 \quad ; \quad B^2 = 1 \quad ; \quad (AB)^2 = \mathbf{1} \tag{E.3}$$

we introduce the following representation of the generators as matrices acting on $\mathbb{C}^3$

$$A = \begin{pmatrix} e^{\frac{2i*\pi}{m}} & 0 & 0 \\ 0 & e^{-\frac{2i*\pi}{m}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.4}$$

$$B = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{E.5}$$

E.2 Abstract structure of the Dihedral Groups Dih_m with $m = \mathrm{odd}$

E.2.1 Conjugacy classes

The total number of conjugacy classes is

$$\ell = \frac{m+3}{2} \tag{E.6}$$

that are enumerated as follows[9]:

$$\begin{aligned}
&\mathscr{C}_1 = \langle E \rangle \\
&\mathscr{C}_2 = \left\langle B, AB, A^2B, \ldots A^{m-1}B \right\rangle, \\
&\mathscr{C}_3 = < A, A^{m-1} >, \mathscr{C}_4 = < A^2, A^{m-2} >, \\
&\mathscr{C}_5 = < A^3, A^{m-3} >, \\
&\ldots, \\
&\mathscr{C}_{\frac{m+3}{2}} = < A^{\frac{m-1}{2}}, A^{\frac{m+1}{2}} >
\end{aligned} \tag{E.7}$$

Hence we have one class of population 1, one class of population m and $\frac{m-1}{2}$ classes of population 2

$$|\mathrm{Dih}_m| = 2m = 1 + m + 2 \times \frac{m-1}{2} \tag{E.8}$$

[9] In table E.7 we denote by E the identity element of the group.

E.2.2 Irreps

Accordingly we expect $\ell = \frac{m+3}{2}$ irreducible representations. They are as follows:

1. The one dimensional identity representation $\mathbb{D}_0$
2. The alternating one-dimensional representation $\mathbb{D}_1$, obtained by setting A $\to$ 1, B $\to$ -1.
3. The $\frac{m-1}{2}$ two-dimensional representations obtained in the following way:

$$\mathbb{D}_{k+1}[A] = \begin{pmatrix} e^{\frac{2i*\pi}{m}k} & 0 \\ 0 & e^{-\frac{2i*\pi}{m}k} \end{pmatrix} \quad ; \quad \mathbb{D}_{k+1}[B] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \left(k, 1, 2, ..., \frac{m-1}{2}\right); \tag{E.9}$$

E.2.3 Characters

In this way we obtain the following character table:

0	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	...	$\mathscr{C}_6$
$\times$	E	B	A	A^2	A^3	...	$A^{\frac{m-1}{2}}$
$\mathbb{D}_1$	1	1	1	1	1	...	1
$\mathbb{D}_2$	1	-1	1	1	1	...	1
$\mathbb{D}_3$	2	0	$2\cos\left[\frac{2\pi}{m}\right]$	$2\cos\left[\frac{4k\pi}{m}\right]$	$2\cos\left[\frac{6k\pi}{m}\right]$	...	$2\cos\left[\frac{(-1+m)\pi}{m}\right]$
$\mathbb{D}_4$	2	0	$2\cos\left[\frac{4\pi}{m}\right]$	$2\cos\left[\frac{8\pi}{m}\right]$	$2\cos\left[\frac{12\pi}{m}\right]$	...	$2\cos\left[\frac{2(-1+m)\pi}{m}\right]$
...	...	...	...	...	...	...	...
$\mathbb{D}_{\frac{m-1}{2}}$	2	0	$2\cos\left[\frac{(-1+m)\pi}{m}\right]$	$2\cos\left[\frac{2(-1+m)\pi}{m}\right]$	$2\cos\left[\frac{3(-1+m)\pi}{m}\right]$	...	$2\cos\left[\frac{(-1+m)^2\pi}{2m}\right]$

(E.10)

E.3 The quiver matrix of Dih_3 acting on $\mathbb{C}^3$

Using the above character table (case $m = 3$) we easily derive the following decomposition into irreps:

$$\mathscr{Q} \otimes \mathbb{D}_i = \bigoplus_{j=1}^{\ell=3} \mathrm{QC_{ij}} \mathbb{D}_j \tag{E.11}$$

where $\mathscr{Q}$ is the three-dimensional representation of the dihedral group defined by eqs. (E.4,E.5) and $\mathbb{D}_i$ are the irreducible representations listed above. The matrix $\mathrm{QC_{ij}}$ is that anticipated in equation (E.1)

E.4 Ages

In a similar easy way we derive the age grading of the three conjugacy classes and the associated triple of integer numbers a_i that define the weights in the weighted blowup procedure. We find:

1. age = 0 ; $1\{0,0,0\}$
2. age = 1 ; $\frac{1}{2}\{1,1,0\}$
3. age = 1 ; $\frac{1}{3}\{0,2,1\}$

Hence apart from the age = 0 class of the identity we find two junior classes and no senior one. In force of the fundamental theorem 41 we conclude that the Hodge numbers of the resolved variety $\mathscr{M}$ are as follows $h^{0,0} = 1$, $h^{1,1} = 2$, $h^{2,2} = 0$. Indeed no (1,1)-class has compact support.

E.5 The regular representation construction and decomposition

The next item that we need is the regular representation of the group Dih_3 and its block diagonalization into irreducible subspaces corresponding to the irreducible representations. As it is well known the regular representation of any finite group contains as many copies of each irrep $\mathbb{D}_i$ as it is its dimension. Hence ordering the three irreducible representations of Dih_3 according to the above presented scheme, namely $\mathbb{D}_1$,$\mathbb{D}_2$, $\mathbb{D}_3$ with

$$\dim \mathbb{D}_1 = 1 \quad ; \quad \dim \mathbb{D}_2 = 1 \quad ; \quad \dim \mathbb{D}_3 = 2 \tag{E.12}$$

and naming R the regular representation we have:

$$R = \mathbb{D}_1 \oplus \mathbb{D}_2 \oplus \mathbb{D}_3 \oplus \widehat{\mathbb{D}}_3 \tag{E.13}$$

With some effort one derives the matrix **m** that performs the change of basis from the natural basis of the regular representation whose axes are the group elements to the block diagonal basis where each block correspond to one irreducible representation. The explicit form of **m** is displayed below:

$$\mathbf{m} = \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 \\ 1 & -1 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) \\ 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \end{pmatrix} \tag{E.14}$$

Using **m** we obtain the explicit form of the 6 group elements of the dihedral group in the block diagonal form of the 6-dimensional regular representation. They are displayed below:

$$\mathrm{R}[1] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{R}[2] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \\ 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \end{pmatrix}$$

$$
\mathrm{R}[3]=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0\\
0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right)\\
0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0
\end{pmatrix}
$$

$$
\mathrm{R}[4]=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

$$
\mathrm{R}[5]=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right)
\end{pmatrix}
$$

$$
\mathrm{R}[6]=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right)
\end{pmatrix}
\tag{E.15}
$$

E.6 The invariant space $\mathscr{S}_\Gamma = \mathbf{Hom}_\Gamma(R, \mathscr{Q} \otimes R)$

Imposing the invariance constraint (6.6) and (5.10) and using the explicit form of the regular representation derived above we obtain the triples of matrices spanning $\mathscr{S}_\Gamma$. They depend on $3 \times 6 = 18$ parameters and their explicit form written in the split basis where the regular representation is block diagonal is displayed below:

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 0 & m_{1,3} & 0 & m_{1,5} & 0 \\ 0 & 0 & m_{2,3} & 0 & m_{2,5} & 0 \\ 0 & 0 & 0 & m_{3,4} & 0 & m_{3,6} \\ m_{4,1} & m_{4,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{5,4} & 0 & m_{5,6} \\ m_{6,1} & m_{6,2} & 0 & 0 & 0 & 0 \end{pmatrix} \\
B &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{1,3} & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{1,5} \\ 0 & 0 & 0 & -\frac{1}{2}\left(i+\sqrt{3}\right) m_{2,3} & 0 & -\frac{1}{2}\left(i+\sqrt{3}\right) m_{2,5} \\ -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{4,1} & -\frac{1}{2}\left(i+\sqrt{3}\right) m_{4,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{3,4} & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{3,6} & 0 \\ -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{6,1} & -\frac{1}{2}\left(i+\sqrt{3}\right) m_{6,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{5,4} & 0 & -\frac{1}{2}\left(-i-\sqrt{3}\right) m_{5,6} & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & r_{1,2} & 0 & 0 & 0 & 0 \\ r_{2,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{3,3} & 0 & r_{3,5} & 0 \\ 0 & 0 & 0 & -r_{3,3} & 0 & -r_{3,5} \\ 0 & 0 & r_{5,3} & 0 & r_{5,5} & 0 \\ 0 & 0 & 0 & -r_{5,3} & 0 & -r_{5,5} \end{pmatrix} \qquad \text{(E.16)}
\end{aligned}$$

E.6.1 The locus $L_\Gamma \subset \mathscr{S}_\Gamma$

The triples of matrices $\{A_0, B_0, C_0\}$ spanning the locus $L_\Gamma \subset \mathscr{S}_\Gamma$ defined in eq. (5.34) depend on three complex parameters z_1,z_2,z_3 that can be regarded as a set of global coordinates for the orbifold locus $\frac{\mathbb{C}^3}{\mathrm{Dih}_3}$. They are displayed below written in the split basis of the regular representation. They are the image after the change of basis of those matrices that belong to $\mathscr{S}_\Gamma$ and are diagonal in the natural basis:

$$
\begin{aligned}
A_0 &= \begin{pmatrix} 0 & 0 & \frac{z_2}{\sqrt{15}} & 0 & \frac{z_1}{\sqrt{15}} & 0 \\ 0 & 0 & -\frac{z_2}{\sqrt{15}} & 0 & \frac{z_1}{\sqrt{15}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2z_2}{\sqrt{15}} \\ \frac{2z_1}{\sqrt{15}} & \frac{2z_1}{\sqrt{15}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2z_1}{\sqrt{15}} & 0 & 0 \\ \frac{2z_2}{\sqrt{15}} & -\frac{2z_2}{\sqrt{15}} & 0 & 0 & 0 & 0 \end{pmatrix} \\
B_0 &= \begin{pmatrix} 0 & 0 & 0 & -\frac{e^{i\frac{7}{6}\pi}z_2}{2\sqrt{15}} & 0 & -\frac{e^{i\frac{7}{6}\pi}z_1}{2\sqrt{15}} \\ 0 & 0 & 0 & -\frac{e^{i\frac{7}{6}\pi}z_2}{2\sqrt{15}} & 0 & \frac{e^{i\frac{7}{6}\pi}z_1}{2\sqrt{15}} \\ -\frac{e^{i\frac{7}{6}\pi}z_1}{\sqrt{15}} & \frac{e^{i\frac{7}{6}\pi}z_1}{\sqrt{15}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{e^{i\frac{7}{6}\pi}z_2}{\sqrt{15}} & 0 \\ -\frac{e^{i\frac{7}{6}\pi}z_2}{\sqrt{15}} & -\frac{e^{i\frac{7}{6}\pi}z_2}{\sqrt{15}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{e^{i\frac{7}{6}\pi}z_1}{\sqrt{15}} & 0 & 0 & 0 \end{pmatrix} \\
C_0 &= \begin{pmatrix} 0 & \frac{z_3}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ \frac{z_3}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{z_3}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{z_3}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{z_3}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{z_3}{\sqrt{6}} \end{pmatrix}
\end{aligned} \tag{E.17}
$$

E.7 The complex quiver group $\mathscr{G}_\Gamma$

Our next point is the derivation of the group $\mathscr{G}_\Gamma$ discussed in section 6.2.1 of the main text and defined by the following condition

$$
\mathscr{G}_\Gamma = \{g \in \mathrm{SL}(|\Gamma|, \mathbb{C}) \quad | \quad \forall \gamma \in \Gamma \,:\, [D_R(\gamma), D_{\mathrm{def}}(g)] = 0\} \tag{E.18}
$$

Abstractly the quiver group turns out to have has the following structure:

$$
\mathscr{G}_\Gamma = \mathbb{C}^* \times \mathbb{C}^* \times \mathrm{SL}(2, \mathbb{C}) \tag{E.19}
$$

Indeed a generic group element of the quiver group depends on 5 complex parameters χ_i (i=1,..,5) and it has the following form:

$$
\mathfrak{g} \in \mathscr{G}_\Gamma \,:\, \mathfrak{g} = \begin{pmatrix} \frac{1}{\chi_1(\chi_3\chi_4 - \chi_2\chi_5)^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \chi_2 & 0 & \chi_3 & 0 \\ 0 & 0 & 0 & \chi_2 & 0 & \chi_3 \\ 0 & 0 & \chi_4 & 0 & \chi_5 & 0 \\ 0 & 0 & 0 & \chi_4 & 0 & \chi_5 \end{pmatrix} \tag{E.20}
$$

E.7.1 Construction of the $\mathbb{V}_{|\Gamma|+2}$ manifold as an orbit of the quiver group

According to the discussion of the main text and with eq. (6.29) we construct the variety $\mathbb{V}_{|\Gamma|+2} = \mathbb{V}_8$ as the orbit $\mathscr{D}_\Gamma = \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$ of the locus L_Γ with respect to the quiver group. Therefore the coordinates on $\mathbb{V}_8$ are $\{z_1, z_2, z_3, \chi_1, \ldots \chi_5\}$.

E.7.2 Reduction to the compact gauge group

The next task is to derive the maximal compact subgroup of the quiver group

$$\mathscr{F}_\Gamma \subset \mathscr{G}_\Gamma \tag{E.21}$$

which, as emphasized several times, is the gauge group of the Chern-Simons theory on the brane volume and the group with respect to which we perform the Kähler quotient. In the present case we have:

$$\mathscr{F}_\Gamma = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2) \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) = \mathscr{G}_\Gamma \tag{E.22}$$

Let us name the corresponding Lie algebras according to the following obvious nomenclature and perform the following orthogonal split

$$\begin{array}{rcl} \mathbb{G}_\Gamma & = & \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma \\ {[\mathbb{F}_\Gamma, \mathbb{F}_\Gamma]} & \subset & \mathbb{F}_\Gamma \quad ; \quad [\mathbb{F}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{K}_\Gamma \quad ; \quad [\mathbb{K}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{F}_\Gamma \end{array} \tag{E.23}$$

The main final item is encoded in the coset manifold:

$$\mathscr{V}_\Gamma \equiv \frac{\mathscr{G}_\Gamma}{\mathscr{F}_\Gamma} \quad ; \quad \dim_{\mathbb{R}} \mathscr{V}_\Gamma = \dim_{\mathbb{C}} \mathscr{G}_\Gamma = \dim_{\mathbb{R}} \mathscr{F}_\Gamma = 5 \quad \text{(in our case)} \tag{E.24}$$

for whose solving element $\exp[\mho]$ we will find a system of algebraic equations encoding the complete resolution of the problem of the Kähler quotient and the definition of all the tautological bundles.

Schematically we will have the following equation. Let $\mathfrak{P}$ represent the moment map

$$\mathfrak{P} : \mathscr{S}_\Gamma \longrightarrow \mathbb{F}_\Gamma \tag{E.25}$$

and let us denote by $g.\mathrm{p}$ the action of the quiver group on the space $\mathscr{S}_\Gamma$:

$$\begin{array}{rcl} \mathscr{G}_\Gamma & : & \mathscr{S}_\Gamma \longrightarrow \mathscr{S}_\Gamma \\ \forall g \in \mathscr{G}_\Gamma \, , \, \Omega_g & : & p \longrightarrow g.p \in \mathscr{S}_\Gamma \end{array} \tag{E.26}$$

The fundamental property of the compact subgroup is the following one:

$$\mathfrak{P}(g.p) = \mathfrak{P}(p) \qquad \text{iff} \qquad g \in \mathscr{F}_\Gamma \subset \mathscr{G}_\Gamma \tag{E.27}$$

In view of this, provided we have chosen some parametrization of the coset $\mathscr{V}_\Gamma$, given by coordinates $\mho_i$ (i=1,..,$\dim_{\mathbb{R}} \mathscr{V}_\Gamma$) we find an equation of the following form:

$$\mathfrak{P}\left(\mathrm{Exp}[\mho].p_0\right) = \zeta \tag{E.28}$$

where

$$\mathfrak{P}\left(p_0\right) = 0 \;\Leftrightarrow\; p_o \in L_\Gamma \tag{E.29}$$

E.7.3 Generators of the Lie Algebra $\mathbb{F}_\Gamma = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2)$

Here we restart our analysis at the level of Lie algebra rather than at the level of Lie Groups and we derive the form of the generators of $\mathscr{F}_\Gamma$. This is very important for the explicit discussion of the Kähler quotient. Then we construct the complementary subspace $\mathbb{K}_\Gamma$.

A generic element of the compact Lie algebra $\mathbb{F}_\Gamma$ can be written as the following 6×6 matrix depending on five real parameters $v_1,...,v_5$:

$$\mathfrak{f} \in \mathbb{F}_\Gamma \; : \; \mathfrak{f} = \begin{pmatrix} i(-v_1 - 2v_2 - 2v_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & iv_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & iv_2 & 0 & v_4 + iv_5 & 0 \\ 0 & 0 & 0 & iv_2 & 0 & v_4 + iv_5 \\ 0 & 0 & -v_4 + iv_5 & 0 & iv_3 & 0 \\ 0 & 0 & 0 & -v_4 + iv_5 & 0 & iv_3 \end{pmatrix} \tag{E.30}$$

Hence the generators are provided by the matrices multiplying each of the parameters $v_1,...,v_5$.

E.7.4 Generators in the complementary subspace $\mathbb{K}_\Gamma$

The explicit form of a matrix belonging to the complementary subspace $\mathbb{K}_\Gamma$ is given here below and depends on the five real parameters ψ_i (i=1,..,5)

$$\mathfrak{k} \in \mathbb{K}_\Gamma \; : \; \mathfrak{k} = \begin{pmatrix} -\psi_1 - 2\psi_2 - 2\psi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_2 & 0 & \psi_4 + i\psi_5 & 0 \\ 0 & 0 & 0 & \psi_2 & 0 & \psi_4 + i\psi_5 \\ 0 & 0 & \psi_4 - i\psi_5 & 0 & \psi_3 & 0 \\ 0 & 0 & 0 & \psi_4 - i\psi_5 & 0 & \psi_3 \end{pmatrix} \tag{E.31}$$

A basis of the subspace $\mathbb{K}_\Gamma$ is given by the matrices that multiply each ψ_i.

The above explicit matrices exemplify the discussion in section 8.2.2 of the main text.

E.7.5 Exponentiation of the orthogonal subspace $\mathbb{K}_\Gamma$

In this subsection we derive the hermitian matrix $\mathscr{V} = \mathscr{V}^\dagger$ = Exp[$\mathbb{K}_\Gamma$] that can be used as a coset representative for the coset $\frac{\mathscr{G}_\Gamma}{\mathscr{F}_\Gamma}$.

One parametrization of the coset manifold $\frac{\mathscr{G}_\Gamma}{\mathscr{F}_\Gamma}$ is provided $\exp[\mathbb{K}_\Gamma] = \mathscr{V}$ which depends on five real parameters, namely the three scale factors $\Upsilon_1, \Upsilon_2, \Upsilon_3$, the hyperbolic angle ρ and the elliptic angle ϕ. Indeed taking the product of the exponentiation of the various generators we get that $\mathscr{V} \in \mathrm{Exp}[\mathbb{K}_\Gamma]$ has the following form:

$$\mathscr{V} = \left(\begin{array}{cccccc} \frac{\mathrm{Cosh}[\rho]^4}{\Upsilon_1\Upsilon_2^2\Upsilon_3^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \Upsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Upsilon_2 & 0 & e^{i\phi}\sqrt{\Upsilon_2\Upsilon_3}\mathrm{Tanh}[\rho] & 0 \\ 0 & 0 & 0 & \Upsilon_2 & 0 & e^{i\phi}\sqrt{\Upsilon_2\Upsilon_3}\mathrm{Tanh}[\rho] \\ 0 & 0 & e^{-i\phi}\sqrt{\Upsilon_2\Upsilon_3}\mathrm{Tanh}[\rho] & 0 & \Upsilon_3 & 0 \\ 0 & 0 & 0 & e^{-i\phi}\sqrt{\Upsilon_2\Upsilon_3}\mathrm{Tanh}[\rho] & 0 & \Upsilon_3 \end{array}\right) \tag{E.32}$$

Subsequently suitably renaming the parameters we can rewrite the above hermitian matrix $\mathscr{V}$ in a different more friendly way which is the following one:

$$\mathscr{V} = \left(\begin{array}{cccccc} \frac{1}{(-1+X^2+Y^2)^2\Upsilon_1\Upsilon_2^2\Upsilon_3^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \Upsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Upsilon_2 & 0 & (X+iY)\sqrt{\Upsilon_2\Upsilon_3} & 0 \\ 0 & 0 & 0 & \Upsilon_2 & 0 & (X+iY)\sqrt{\Upsilon_2\Upsilon_3} \\ 0 & 0 & (X-iY)\sqrt{\Upsilon_2\Upsilon_3} & 0 & \Upsilon_3 & 0 \\ 0 & 0 & 0 & (X-iY)\sqrt{\Upsilon_2\Upsilon_3} & 0 & \Upsilon_3 \end{array}\right) \tag{E.33}$$

The 5 real parameters $\Upsilon_1, \Upsilon_2, \Upsilon_3 > 0$ and X, Y play a fundamental role in the elaboration of the tautological bundles.

E.7.6 Calculation of the center of the compact Lie algebra

We are interested in the calculation of the center of the compact Lie algebra $\mathfrak{z}[\mathbb{F}_\Gamma]$ since it is only the moment maps in this center that can be assigned non vanishing values. The abstract form of the searched for center is the following:

$$\mathfrak{z}[\mathbb{F}_\Gamma] = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \tag{E.34}$$

To find the explicit immersion of $\mathfrak{z}[\mathbb{F}_\Gamma]$ in $\mathbb{F}_\Gamma$ we impose the condition that an element of the center should commute with all the generators of $\mathbb{F}_\Gamma$. In this way we can reorganize the listing of the 5 generators into 2 belonging to the center for which we can lift the level of the moment map from 0 to a finite value ζ and 3 whose moment map must remain at level 0. Henceforth we introduce a new basis for the complete Lie Algebra $\mathbb{F}_\Gamma$ which is reorganized as follows:

The two central generators

$$\mathfrak{z}_1=\begin{pmatrix} -i\sqrt{\frac{5}{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{30}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{30}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{30}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{30}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{30}} \end{pmatrix} \; ; \; \mathfrak{z}_2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2i}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2\sqrt{5}} \end{pmatrix} \tag{E.35}$$

The three generators of $\mathfrak{su}(2)$

$$J_1=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} \end{pmatrix} ; J_2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} ; J_3=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \end{pmatrix} \tag{E.36}$$

that satisfy the standard commutation relations:

$$[J_i\,,J_j]\,=\,\varepsilon_{ijk}\,J_k \tag{E.37}$$

According to the general discussion of section 8.2.2 as a basis of $\mathbb{K}_\Gamma$ we can just take the same 5 generators listed above, each multiplied by an i factor.

E.8 Fixing the zero level point and the moment maps

In the previously analyzed abelian cases the point in the $\mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$ where the moment map vanishes and which therefore corresponds to the orbifold, was just the locus L_Γ itself, defined as that one where, in the natural basis of the Regular Representation the three matrices A,B,C are diagonal. There is nothing magic in that point apart from the fact of being a convenient starting point to calculate the entire orbit. What has an intrinsic geometrical significance in the $\mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma) = \mathbb{V}_{|\Gamma|+2}$ is the point p_0 where the moment map vanishes for all components. This is the orbifold limit $\frac{\mathbb{C}^3}{\Gamma} \Leftrightarrow p^0_\Gamma \in \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$

$$\mathfrak{P}\left(p^0_\Gamma\right) = 0 \tag{E.38}$$

The present section is devoted to the determination of p^0_Γ=$\{A_0$,B_0,$C_0\}$.

E.8.1 Construction of the moment maps

In order to construct the moment maps we need the matrices {A, B, C} in the orbit and their hermitian conjugate $A^\dagger$, $B^\dagger$, $C^\dagger$. Then the Kähler potential of the ambient space is defined as follows:

$$\mathscr{K} \;= \mathrm{Tr}\left(A^\dagger A\right) + \;\mathrm{Tr}\left(B^\dagger B\right) + \mathrm{Tr}\left(C^\dagger C\right) \tag{E.39}$$

while the moment maps are obtained from

$$\mathfrak{P}_I = -i\left(\mathrm{Tr}\left(T_I\left[A,A^\dagger\right]\right) + \mathrm{Tr}\left(T_I\left[B,B^\dagger\right]\right) + \mathrm{Tr}\left(T_I\left[C,C^\dagger\right]\right)\right)\ ;\ (I = 1,...,5) \tag{E.40}$$

where T_I are the generators of the compact subgroup.

Calculating the moment maps at p $\in L_\Gamma$ we find that they are not zero, yet it suffices to apply a transformation of exp[$\mathbb{K}_\Gamma$] as in equation (E.33) with the following parameters:

$$\Upsilon_1 = \left(\sqrt[6]{\frac{1}{4}}\right), \quad \Upsilon_2 = \sqrt[6]{2}, \quad \Upsilon_3 - \frac{1}{\left(\sqrt[6]{\frac{1}{4}}\right) * \sqrt[6]{2}}, \quad \rho = 0, \quad \phi = 0 \tag{E.41}$$

and the moment map vanishrs. In this way we have found the point $p^0_\Gamma \in \mathrm{Orbit}_{\mathscr{G}_\Gamma}(L_\Gamma)$ mentioned above which exactly corresponds to the orbifold limit $\frac{\mathbb{C}^3}{\Gamma}$. The explicit form of the triple p^0_Γ={A^0,B^0,C^0} is displayed below (where $\xi = \frac{7}{6}\pi$):

$$\begin{aligned}
A^0 &= \begin{pmatrix}
0 & 0 & \sqrt{\frac{2}{15}}z_2 & 0 & \sqrt{\frac{2}{15}}z_1 & 0 \\
0 & 0 & -\sqrt{\frac{2}{15}}z_2 & 0 & \sqrt{\frac{2}{15}}z_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2z_2}{\sqrt{15}} \\
\sqrt{\frac{2}{15}}z_1 & \sqrt{\frac{2}{15}}z_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2z_1}{\sqrt{15}} & 0 & 0 \\
\sqrt{\frac{2}{15}}z_2 & -\sqrt{\frac{2}{15}}z_2 & 0 & 0 & 0 & 0
\end{pmatrix} \\
B^0 &= \begin{pmatrix}
0 & 0 & 0 & -\frac{e^{i\xi}z_2}{\sqrt{30}} & 0 & -\frac{e^{i\xi}z_1}{\sqrt{30}} \\
0 & 0 & 0 & -\frac{e^{i\xi}z_2}{\sqrt{30}} & 0 & \frac{e^{i\xi}z_1}{\sqrt{30}} \\
-\frac{e^{i\xi}z_1}{\sqrt{30}} & \frac{e^{i\xi}z_1}{\sqrt{30}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{e^{i\xi}z_2}{\sqrt{15}} & 0 \\
-\frac{e^{i\xi}z_2}{\sqrt{30}} & -\frac{e^{i\xi}z_2}{\sqrt{30}} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{e^{i\xi}z_1}{\sqrt{15}} & 0 & 0 & 0
\end{pmatrix} \\
C^0 &= \begin{pmatrix}
0 & \frac{z_3}{\sqrt{6}} & 0 & 0 & 0 & 0 \\
\frac{z_3}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{z_3}{\sqrt{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{z_3}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{z_3}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{z_3}{\sqrt{6}}
\end{pmatrix}
\end{aligned} \tag{E.42}$$

The above matrices depend on new triple of complex parameters z_i that will eventually be interpreted as the complex coordinates of the resolved variety just as they are the coordinates of the locus $\frac{\mathbb{C}^3}{\Gamma}$ when the moment map is not lifted from its zero value. Note that we have named A^0, B^0, C^0 the matrices at the point p_0 in order to distinguish them from the matrices A_0, B_0, C_0 that correspond to the locus L_Γ.

E.8.2 Construction of the moment maps starting from the right zero-point

Next we reconsider the orbit starting from the triple of matrices that have zero-moment map:

$$p_0 = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \quad ; \quad p = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \mathscr{V}^{-1} A_0 \mathscr{V} \\ \mathscr{V}^{-1} B_0 \mathscr{V} \\ \mathscr{V}^{-1} C_0 \mathscr{V} \end{pmatrix} \tag{E.43}$$

and we also utilize a polar parametrization of the coordinates

$$z_i = \Delta_i \,\mathrm{Exp}\,[i\,\theta_i] \tag{E.44}$$

Inserting the newly calculated matrices in the formula (E.39) for the Kähler potential we obtain a complicated Kähler potential that for p = p_0 reduces to:

$$\mathscr{K}_0 = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \sum_{i=1}^{3} |z_i|^2 \tag{E.45}$$

This shows that at zero level the space is indeed every where flat except for the singular fixed point.

Next we construct the explicit form of the moment maps for a generic point of the orbit and we obtain five algebraic functions $\mathfrak{P}_I(\Upsilon, X, Y)$ of the five parameters $\Upsilon_{1,2,3}$,X, Y that depend on the three moduli $\Delta_{1,2,3}$ and the three phases $\theta_{1,2,3}$ of the complex coordinates. Equating the five moment maps to $\{\zeta_1, \zeta_2, 0, 0, 0\}$ we have an algebraic system that determines the parameters Υ, X, Y in terms of the levels $\zeta_{1,2}$ and the coordinates z_i.

Higher degree algebraic system. The explicit expressions of the moment maps are rather formidable and it is difficult to display them on paper since they are very large. Furthermore the degree of the equations is certainly higher than the fourth and there is no hope to solve them by radicals. Yet we know that there is just one and only one solution that has the correct reality property, namely $\Upsilon_{1,2,3}$,X, Y are all real and $\Upsilon_{1,2,3}$ are also positive. A convenient way to verify such property of the system is provided by considering small deformations, namely level parameters $\zeta_{1,2}$ infinitesimally close to zero.

First order solution of the algebraic equations We are supposed to solve the moment map equations. We recall that the last three moment maps have to be zero, while the first two have to be lifted to an arbitrary level ζ. We consider the solution by power series in the neighborhood of the identity element, namely we set:

$$\Upsilon_1 \to 1 + \varepsilon\omega_1, \quad \Upsilon_2 \to 1 + \varepsilon\omega_2, \quad \Upsilon_3 \to 1 + \varepsilon\omega_3, \quad X \to x\varepsilon, \quad Y \to y\varepsilon \tag{E.46}$$

and

$$\zeta_1 \to \varepsilon c \quad , \quad \zeta_2 \to \varepsilon d \tag{E.47}$$

where ε is an infinitesimal parameter. At first order in ε the moment maps are:

$$
\begin{aligned}
\mathfrak{P}_1 &= \varepsilon\left(-\frac{4}{3}i\left(2\left(x\mathrm{Cos}\left[\theta_1-\theta_2\right]-y\mathrm{Sin}\left[\theta_1-\theta_2\right]\right)\Delta_1\Delta_2\right.\right. \\
&\quad \left.\left.+\Delta_1^2\left(\omega_1+\omega_2+\omega_3\right)+\left(\Delta_2^2+2\Delta_3^2\right)\left(\omega_1+\omega_2+\omega_3\right)\right)\right)+\mathscr{O}\left(\varepsilon^2\right) \\
\mathfrak{P}_2 &= \varepsilon\left(-\frac{2}{3}i\left(8\left(x\mathrm{Cos}\left[\theta_1-\theta_2\right]-y\mathrm{Sin}\left[\theta_1-\theta_2\right]\right)\Delta_1\Delta_2+\right.\right. \\
&\quad \left.\left.8\Delta_3^2\left(\omega_1+\omega_2+\omega_3\right)+\Delta_1^2\left(4\omega_1+13\left(\omega_2+\omega_3\right)\right)+\Delta_2^2\left(4\omega_1+13\left(\omega_2+\omega_3\right)\right)\right)\right)+\mathscr{O}\left(\varepsilon^2\right) \\
\mathfrak{P}_3 &= \varepsilon\left(-i\left(\Delta_1^2+\Delta_2^2\right)\left(\omega_2-\omega_3\right)\right)+\mathscr{O}\left(\varepsilon^2\right) \\
\mathfrak{P}_4 &= \varepsilon\left(-\frac{4}{3}i\left(y\Delta_1^2+y\left(\Delta_2^2+2\Delta_3^2\right)+\Delta_1\Delta_2\left(y\mathrm{Cos}\left[\theta_1-\theta_2\right]-x\mathrm{Sin}\left[\theta_1-\theta_2\right]\right.\right.\right. \\
&\quad \left.\left.\left.-\mathrm{Sin}\left[\theta_1-\theta_2\right]\omega_1-\mathrm{Sin}\left[\theta_1-\theta_2\right]\omega_2-\mathrm{Sin}\left[\theta_1-\theta_2\right]\omega_3\right)\right)\right)+\mathscr{O}\left(\varepsilon^2\right) \\
\mathfrak{P}_5 &= \varepsilon\left(-\frac{4}{3}i\left(x\Delta_1^2+x\left(\Delta_2^2+2\Delta_3^2\right)-\Delta_1\Delta_2\left(x\mathrm{Cos}\left[\theta_1-\theta_2\right]+y\mathrm{Sin}\left[\theta_1-\theta_2\right]-\mathrm{Cos}\left[\theta_1-\theta_2\right]\omega_1\right.\right.\right. \\
&\quad \left.\left.\left.-\mathrm{Cos}\left[\theta_1-\theta_2\right]\omega_2-\mathrm{Cos}\left[\theta_1-\theta_2\right]\omega_3\right)\right)\right)+\mathscr{O}\left(\varepsilon^2\right)
\end{aligned}
\tag{E.48}
$$

Equating the 5-vector of these moment maps to the 5-vector {c,d,0,0,0} we obtain the solution:

$$
\begin{aligned}
\omega_1 &\to \frac{1}{12}i\left(\frac{4c-2d}{\Delta_1^2+\Delta_2^2}+\frac{9c\left(\Delta_1^4+\left(\Delta_2^2+2\Delta_3^2\right)^2+\Delta_1^2\left(\Delta_2^2+4\Delta_3^2\right)\right)}{\Delta_1^6-2\mathrm{Cos}\left[3\left(\theta_1-\theta_2\right)\right]\Delta_1^3\Delta_2^3+6\Delta_1^4\Delta_3^2+6\Delta_1^2\Delta_3^2\left(\Delta_2^2+2\Delta_3^2\right)+\left(\Delta_2^2+2\Delta_3^2\right)^3}\right) \\
\omega_2 &\to -\frac{i(2c-d)}{12\left(\Delta_1^2+\Delta_2^2\right)} \\
\omega_3 &\to -\frac{i(2c-d)}{12\left(\Delta_1^2+\Delta_2^2\right)} \\
x &\to -\frac{3ic\Delta_1\Delta_2\left(\mathrm{Cos}\left[\theta_1-\theta_2\right]\Delta_1^2+\mathrm{Cos}\left[2\left(\theta_1-\theta_2\right)\right]\Delta_1\Delta_2+\mathrm{Cos}\left[\theta_1-\theta_2\right]\left(\Delta_2^2+2\Delta_3^2\right)\right)}{4\left(\Delta_1^6-2\mathrm{Cos}\left[3\left(\theta_1-\theta_2\right)\right]\Delta_1^3\Delta_2^3+6\Delta_1^4\Delta_3^2+6\Delta_1^2\Delta_3^2\left(\Delta_2^2+2\Delta_3^2\right)+\left(\Delta_2^2+2\Delta_3^2\right)^3\right)} \\
y &\to \left(3ic\mathrm{Sin}\left[\theta_1-\theta_2\right]\Delta_1\Delta_2\right)\times\left(4\left(\Delta_1^4+2\mathrm{Cos}\left[\theta_1-\theta_2\right]\Delta_1^3\Delta_2+2\mathrm{Cos}\left[\theta_1-\theta_2\right]\Delta_1\Delta_2\left(\Delta_2^2+2\Delta_3^2\right)\right.\right. \\
&\quad \left.\left.+\left(\Delta_2^2+2\Delta_3^2\right)^2+\Delta_1^2\left(\left(1+2\mathrm{Cos}\left[2\left(\theta_1-\theta_2\right)\right]\right)\Delta_2^2+4\Delta_3^2\right)\right)\right)^{-1}
\end{aligned}
\tag{E.49}
$$

This clearly shows that the 5-fields $\Upsilon_1,\Upsilon_2,\Upsilon_3,X,Y$ are all activated and equally necessary in the solution of the moment map equation as soon as we move out of level zero.

E.9 The tautological bundles and their Chern classes

Assuming that we have solved the algebraic equations for the fields $\Upsilon_1,\Upsilon_2,\Upsilon_3,X,Y$ (at first order in the level parameters we have done it, and for many considerations this might turn out to be sufficient) we can now utilize the present example in order to illustrate the discussion of the main text in section 8.2.1. With reference to the matrix $\mathscr{H}$ in eq. (8.27) we have determined it. Indeed in our case of Dih_3 there are two non trivial irreducible representations and correspondingly two tautological bundles, respectively of rank 1 and of rank 2. The matrix

$\mathscr{H}$ has the following appearance:

$$\mathscr{H} = \begin{pmatrix} \Upsilon_1 & 0 & 0 \\ 0 & \Upsilon_2 & (X - iY)\sqrt{\Upsilon_2 \Upsilon_3} \\ 0 & (X + iY)\sqrt{\Upsilon_2 \Upsilon_3} & \Upsilon_3 \end{pmatrix} \tag{E.50}$$

which corresponds to

$$\mathfrak{H}_1 = \Upsilon_1 \quad ; \quad \mathfrak{H}_2 = \begin{pmatrix} \Upsilon_2 & (X - iY)\sqrt{\Upsilon_2 \Upsilon_3} \\ (X + iY)\sqrt{\Upsilon_2 \Upsilon_3} & \Upsilon_3 \end{pmatrix} \tag{E.51}$$

and we get

$$\mathrm{Det}\,[\mathfrak{H}_2] = \Upsilon_2 \Upsilon_3 \left(1 - X^2 - Y^2\right) \tag{E.52}$$

This allows us to calculate the first Chern classes explicitly according to eq. (8.31) and the Kähler potential according to eq. (8.32)

The provided illustration of the present nonabelian case, which is the smallest possible one, was finalized to show that everything in the Kronheimer-like construction is fully algorithmic and uniquely determined. The bottleneck however is localized in the system of algebraic equations for the entries of the matrix $\mathscr{H}$ that, also in the simplest cases, are typically quite formidable and of higher degree. Yet it appears and it is worth further investigation that the relevant topological information is fully encapsulated in the first order approximation of small level parameters ζ. We shall go back to this issue in future publications.

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